# Cosmo$\mathcal{L}$attice school:

## Lesson 2a: Introduction to Inflation and post-inflationary dynamics

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# CosmoLattice – School 2022

**Day 1**
(Monday 5th)

- Lesson 1: What is a Lattice? ✓
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

**Day 2**
(Tuesday 6th)

- Lesson 3: Evolution algorithms ODE
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

**Day 3**
(Wednesday 7th)

- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

**Day 4**
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

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LESSON 2a:

- ➤ Introduction to Friedmann equations and inflation
- ➤ Non-linear field dynamics after inflation
  - Example: Preheating in $\lambda\varphi^4$ potential

LESSON 2b:

- ➤ Introduction to CosmoLattice
- ➤ Lattice simulation of preheating in λφ4 potential

PRACTICE:

- ➤ Compilation and execution of CosmoLattice

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# Cosmological principle

- **GENERAL RELATIVITY:**

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

Stress-energy tensor

**Einstein field eqs.**

**GEOMETRY** ⟷ **MATTER**

- **Cosmological Principle:** the Universe is <u>homogeneous</u> and <u>isotropic</u> at large scales.

**GEOMETRY:**

$$\boxed{ds^2 \equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -dt^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]} \longrightarrow \boxed{H(t) \equiv \frac{\dot{a}}{a}}$$

Scale factor

Spatial curvature

**FLRW metric**

**Hubble parameter**

**MATTER:**

$$\boxed{T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}}$$

**Stress-energy tensor of perfect fluid**

characterized by
- $\rho(t)$ : Energy density
- $p(t)$ : pressure density
- $u^\mu$ : 4-velocity; $u_\mu u^\mu = -1 \xrightarrow{\text{comoving observer}} u^\mu = (-1,0,0,0)$

# Friedmann equations

- **Friedmann equations** (General relativity + cosmological principle):

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3m_p^2} + \frac{\Lambda}{3} - \frac{k}{a^2}}$$

1st Friedmann equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{-1}{6m_p^2}(\rho + 3p) + \frac{\Lambda}{3}}$$

2nd Friedmann equation

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0}$$

Conservation equation (not independent)

$$m_p = (8\pi G)^{-1/2}$$

Note: We set from now on $k = \Lambda = 0$

- **Equation of state:**

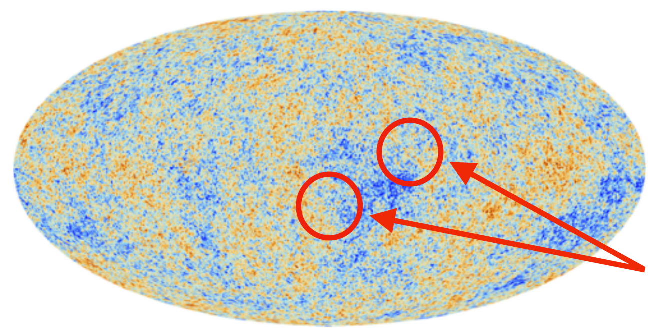
$$\boxed{w \equiv \frac{p}{\rho} = const} \longrightarrow \begin{cases} a(t) \propto t^{\frac{2}{3(1+w)}} & if \;\; w > -1 \\ a(t) \propto e^{Ht} & if \;\; w = -1 \end{cases} \longrightarrow \boxed{\text{If } w>-1/3\text{:} \;\; \frac{d^2a}{dt^2} < 0}$$

- **If w > -1/3, the universe decelerates**. This is the case for SM particles, which can be described as radiation (w=1/3) or matter (w=0).

# Inflation

- A decelerating universe gives rise to the so-known “problems” of classical cosmology:

  - **Horizon problem:**



$$T_{\rm CMB} = 2.72548 \pm 0.00057 K$$

causally disconnected!

  - **Flatness problem:** $\Omega_k \equiv -k/(a_0^2 H_0^2) = -0.0106 \pm 0.0065 \ll 1$

but $\Omega_k$~0 is point of unstable equilibrium!

**Inflation**: An early stage of accelerated expansion of the universe.

$$\boxed{\frac{d^2a}{dt^2} > 0} \longleftrightarrow \boxed{w < -1/3}$$

Inflation solves the horizon and flatness problems. It also generates primordial fluctuations that allow the later structure formation.

Refs: Starobinsky, Guth, Linde (1980-1982)

# Scalar field in a FLRW metric

**Inflation can be sourced by the vacuum energy of a scalar field in "slow-roll"**

- **Action of a scalar field:** $\phi \equiv \phi(t, \vec{x})$

$$S = \int d^4x \sqrt{-g}\,\mathscr{L}_M \qquad \boxed{-\mathscr{L}_M = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)} \qquad g = \det(g_{\mu\nu})$$

- **Stress-energy tensor:**

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathscr{L}_m)}{\delta g_{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi$$

P. fluid

$$\rho_\phi \equiv T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

$$p_\phi \equiv \frac{1}{3}\sum_i T^i_i = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\nabla\phi)^2 - V(\phi)$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_\phi}{3m_p^2}} \qquad \boxed{\frac{\ddot{a}}{a} = -\frac{1}{6m_p^2}(\rho_\phi + 3p_\phi)}$$

- **Equation of motion:**

$$\frac{\delta S}{\delta\phi} = 0 \;\longrightarrow\; \boxed{\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3\frac{\dot{a}}{a}\dot{\phi} + V_{,\phi} = 0}$$

Scalar field EOM

# Slow-roll inflation

- We need to achieve an accelerated universe (w<-1/3): $\phi = \phi(t)$

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\cancel{\frac{1}{2}\dot\phi^2} - V(\phi)}{\cancel{\frac{1}{2}\dot\phi^2} + V(\phi)} \simeq -1 \quad \longrightarrow \quad \dot\phi^2 \ll |V(\phi)|$$

- This stage must take long enough, so the inflaton must not accelerate:

$$\cancel{\ddot\phi} + 3H\dot\phi + V_{,\phi}(\phi) = 0 \quad \longrightarrow \quad |\ddot\phi| \ll 3H|\dot\phi|, V_{,\phi}(\phi)$$

- These two conditions can be written in terms of slow-roll parameters:

$$\boxed{\begin{array}{l} \epsilon \equiv \dfrac{3\dot\phi^2}{2V(\phi)} \ll 1 \\[2ex] \eta \equiv -\dfrac{\ddot\phi}{H\dot\phi} \ll 1 \end{array}} \quad \longrightarrow \quad \boxed{\begin{array}{l} \epsilon_V \equiv \dfrac{m_p^2}{2}\left(\dfrac{V'(\phi)}{V(\phi)}\right)^2 \ll 1 \\[2ex] \eta_V \equiv m_p^2 \left(\dfrac{V''(\phi)}{V(\phi)}\right) \ll 1 \end{array}}$$

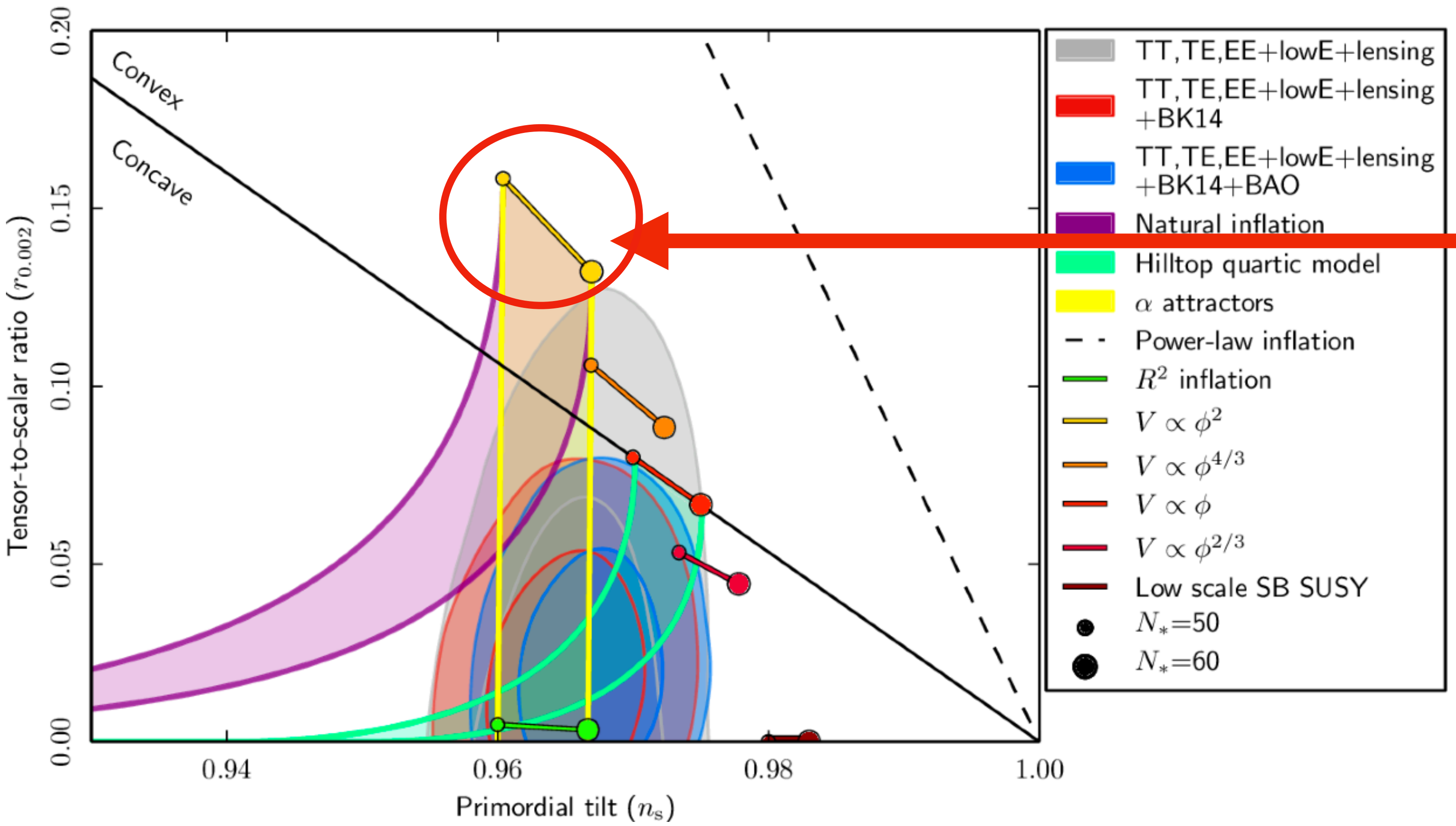
$$\epsilon \simeq \epsilon_V$$
$$\eta \simeq \eta_V - \epsilon_V$$

- Inflation ends when: $\epsilon \simeq \epsilon_V \approx 1$

# Inflation: primordial perturbations

➤ Slow-roll inflation generates an **(almost) scale-invariant spectrum of scalar and tensor perturbations.**

➤ Different V(ϕ) give rise to small differences, which can be parametrized (at first order) by the **scalar tilt $n_s$** and **tensor-to-scalar ratio r**.




**Planck (2018)**

Monomial potentials are ruled out by observations of the CMB anisotropies

"Flat" potentials are favoured

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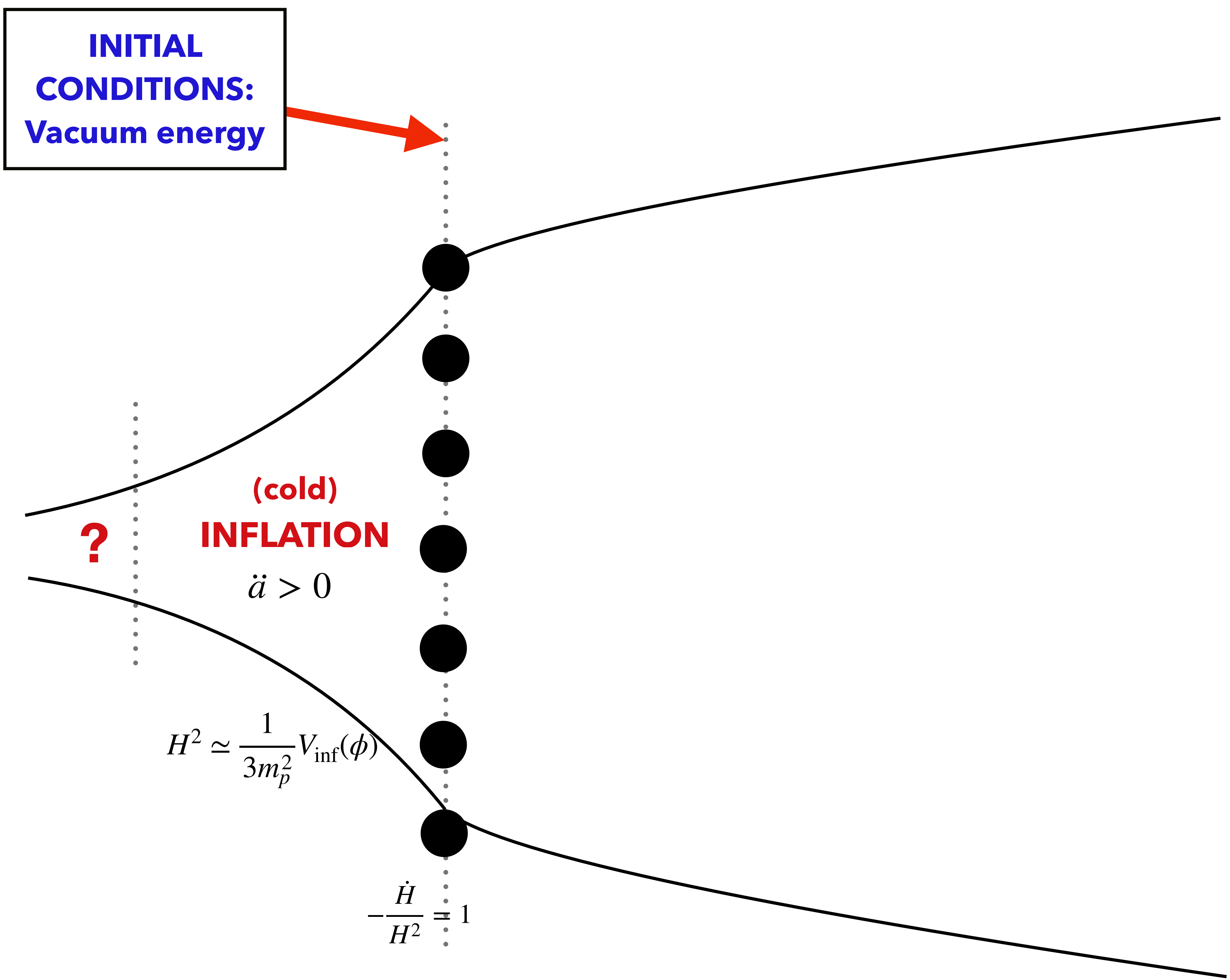
# History of the early universe



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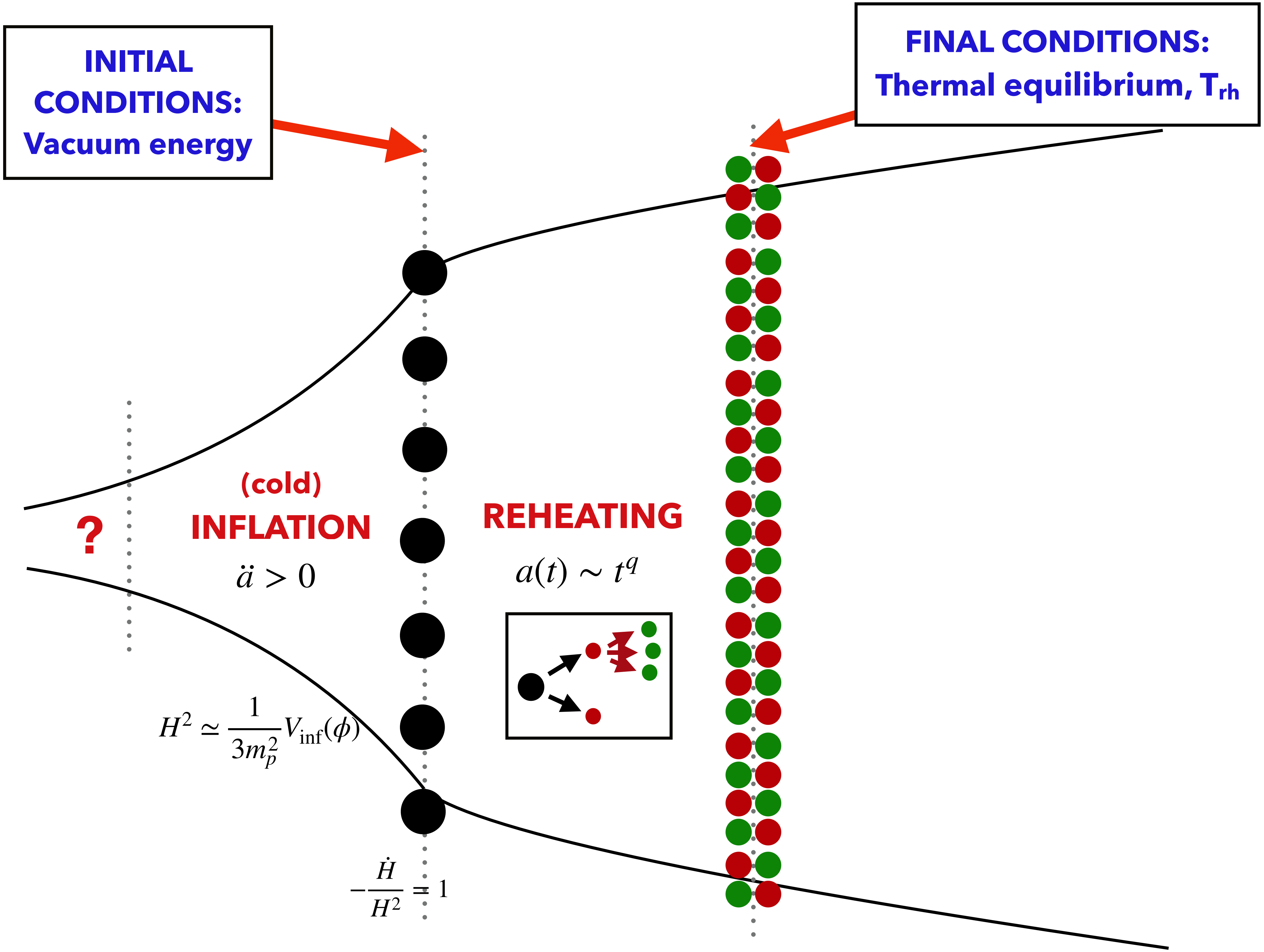
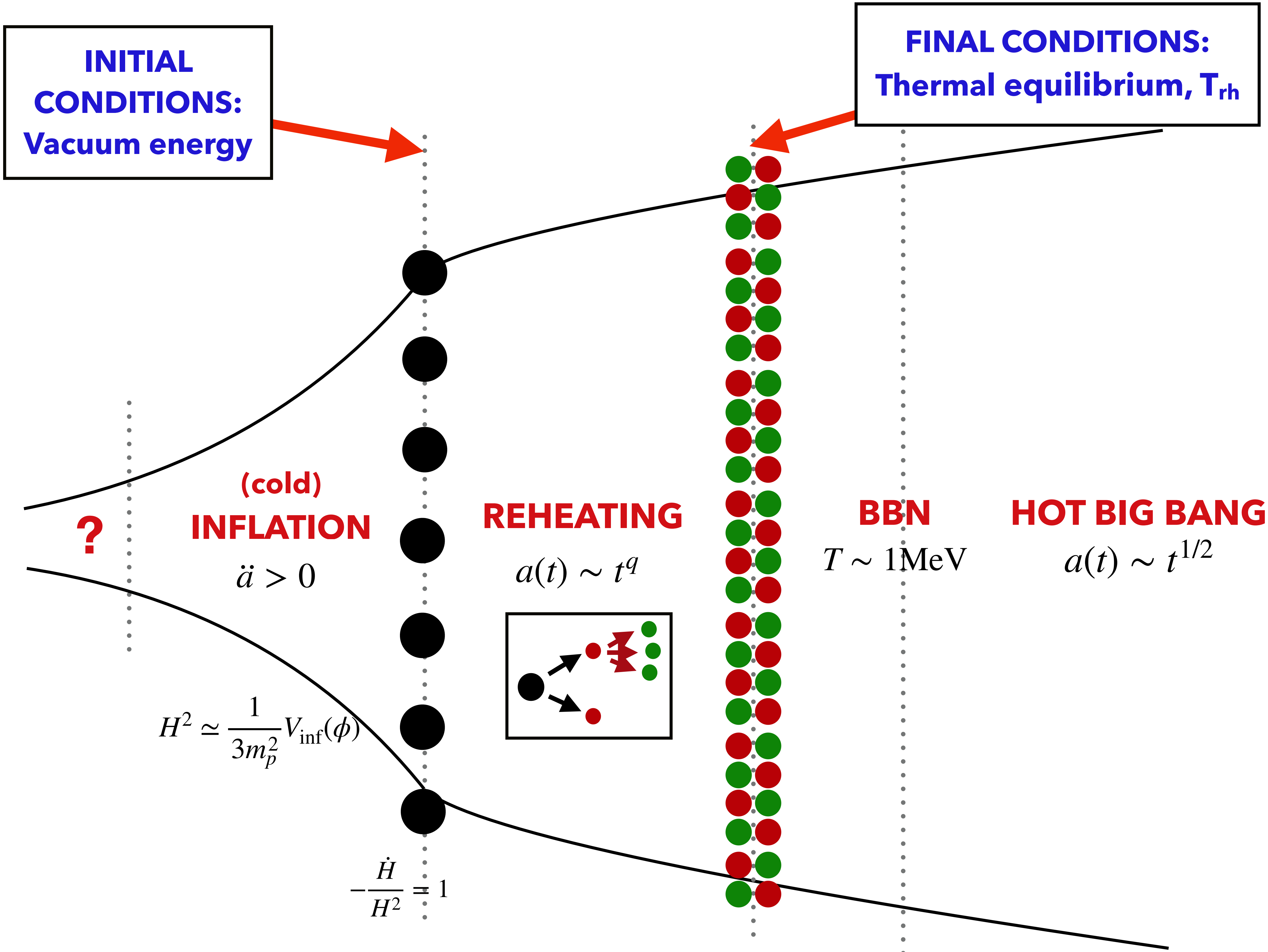
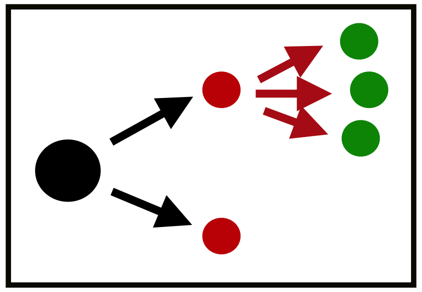


# History of the early universe



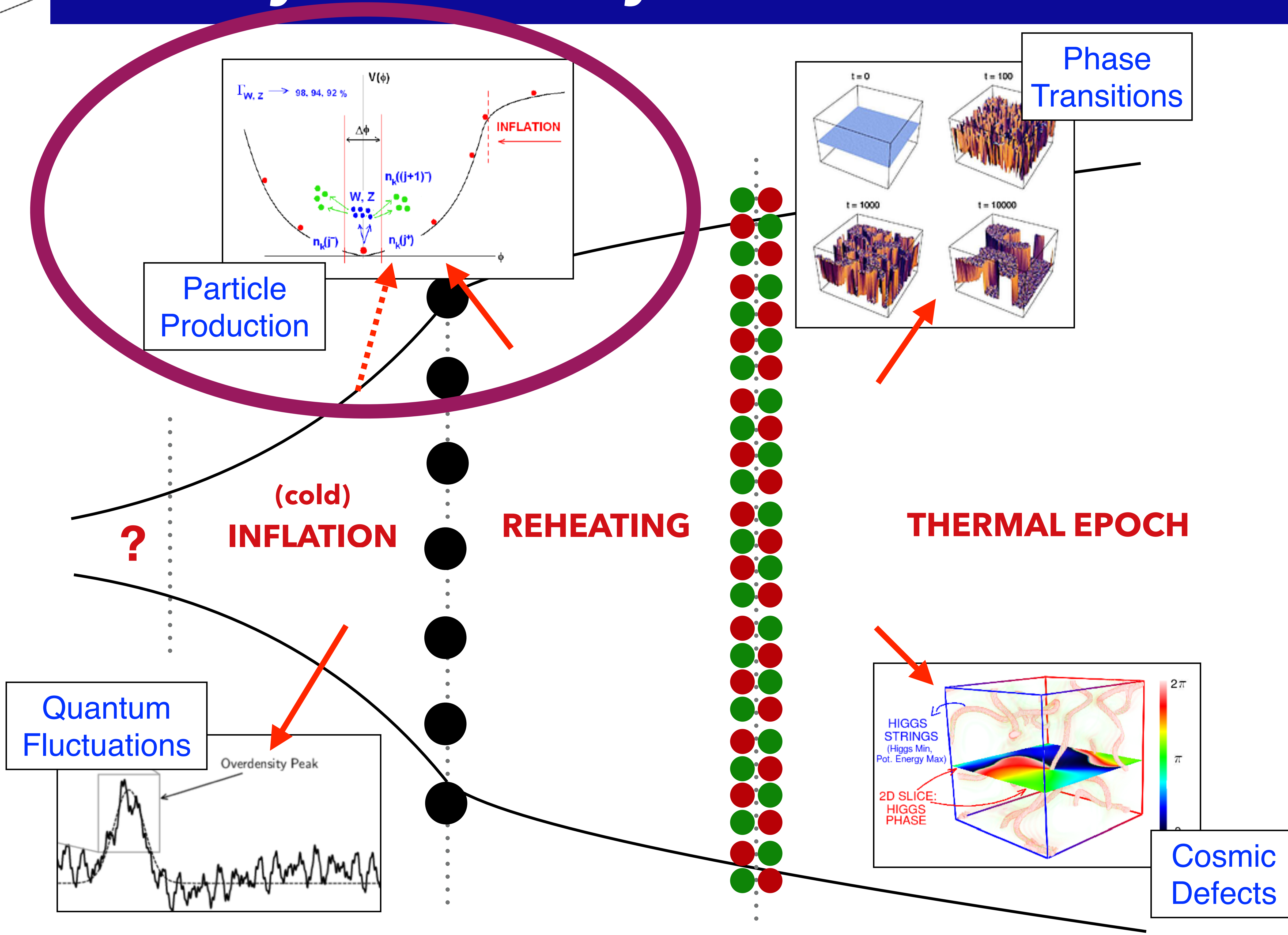
# Non-linear dynamics of the early universe



$$\mathscr{L} = \mathscr{L}(\phi, \varphi_i, \psi_j, A_\mu, h_{\mu\nu}, \ldots)\textbf{??}$$

- Details of the early universe evolution depends on the high-energy physics model.
- Non-linear, non-perturbative, non-equilibrium physics.

# History of the early universe



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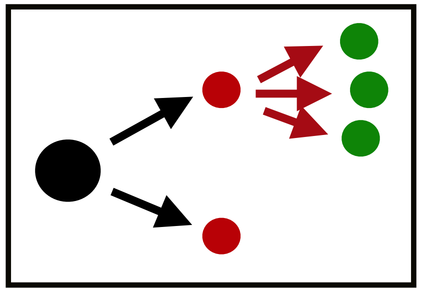


$$\mathscr{L} = \mathscr{L}(\phi, \varphi_i, \psi_j, A_\mu, h_{\mu\nu}, \ldots)$$ **??**

- Details of the early universe evolution depends on the high-energy physics model.
- Non-linear, non-perturbative, non-equilibrium physics.
- EXAMPLE: PREHEATING: field instabilities due to non-perturbative effects.

Classical regime:

$$n_k \sim |X_k|^2 \gg 1$$ → **(classical) LATTICE SIMULATIONS**

- Lattice simulation can be used to study: Inflation, (p)reheating, GWs, cosmic strings and other topological defects, phase transitions, oscillons,…

# Monomial inflaton potentials

- Monomial potentials are ruled out during inflation. However, many potentials behave as a monomial around the minimum.

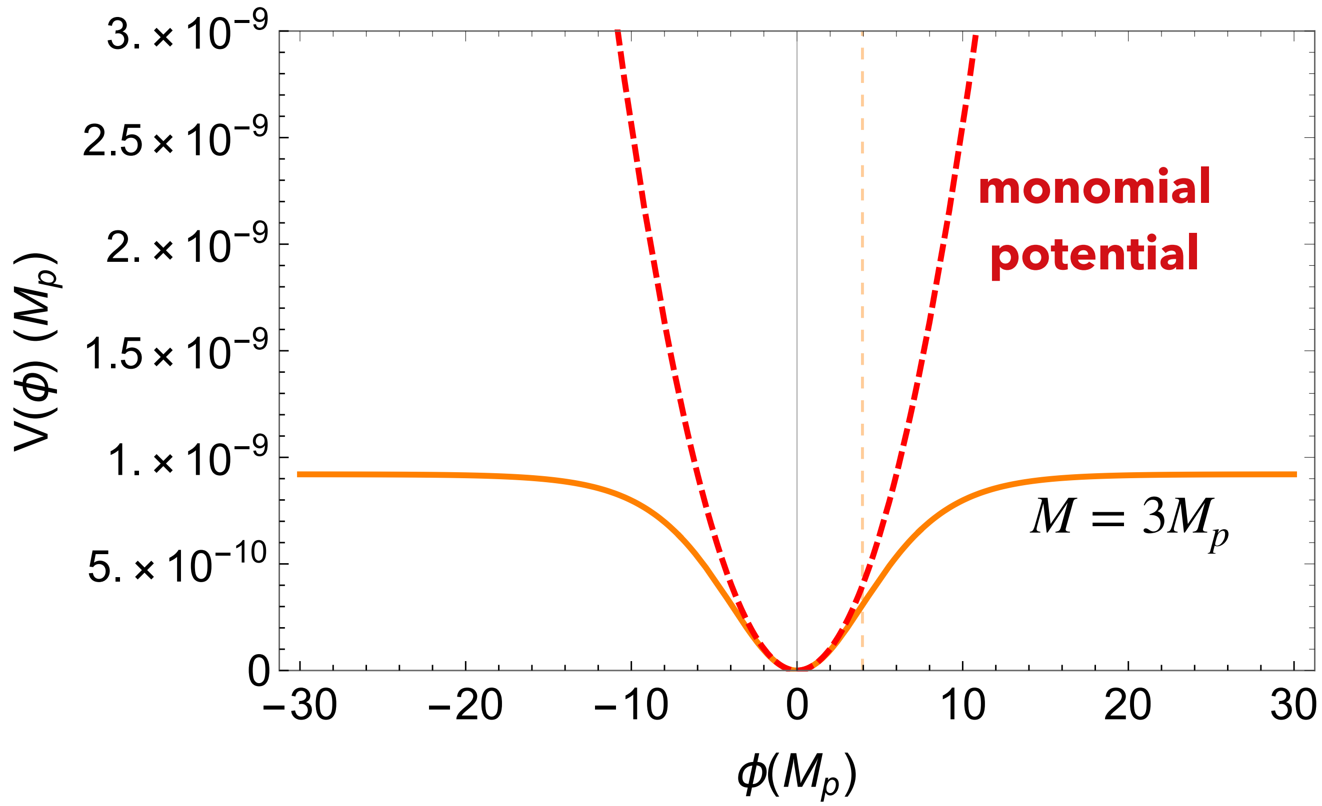
**Example:** alpha-attractor T-models

Kallosh & Linde, JCAP (2013)

$$\boxed{V(\phi) = \frac{\Lambda^4}{p}\tanh^p\left(\frac{|\phi|}{M}\right)}$$

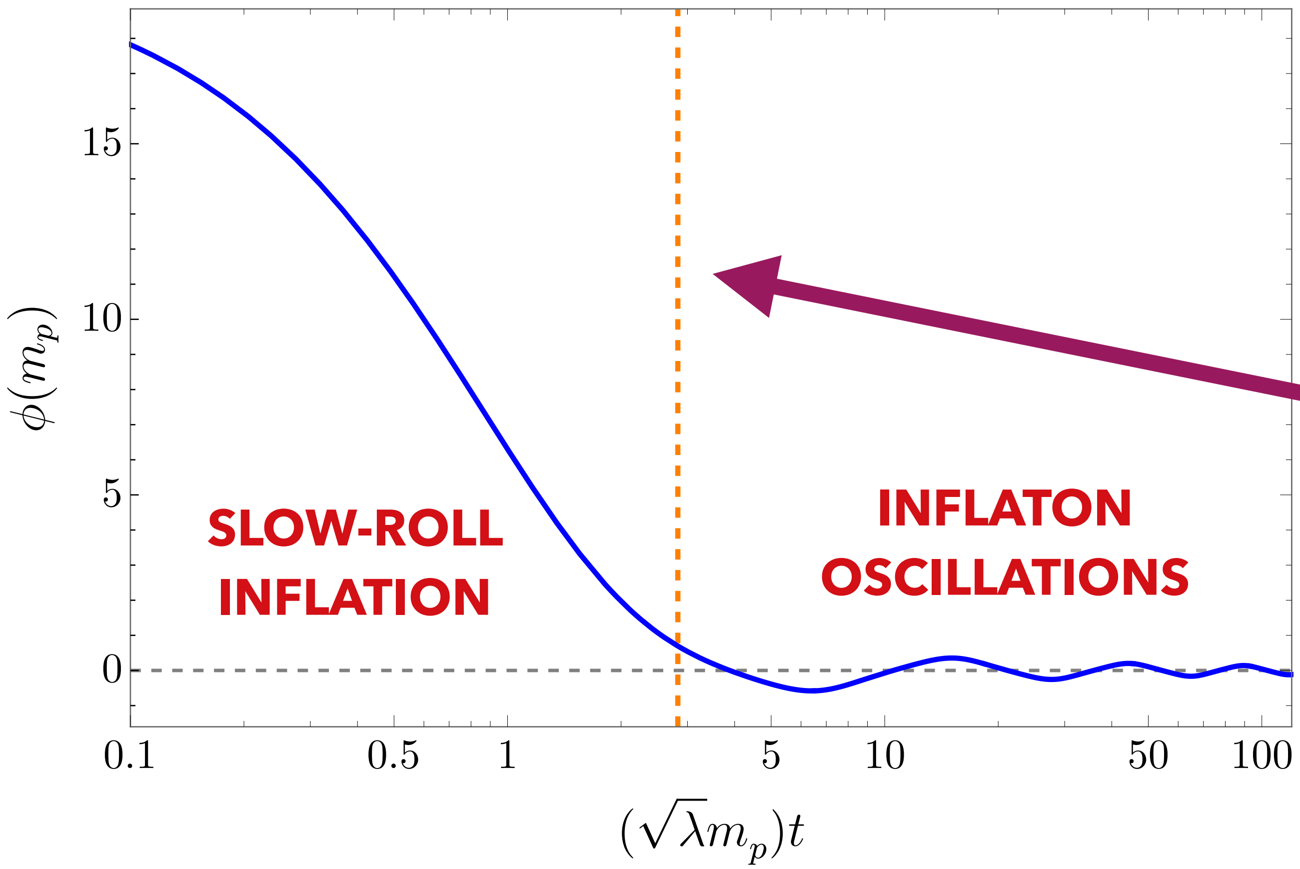
$$\Downarrow$$

$$V(\phi) = \frac{\Lambda^4}{pM^p}|\phi|^p + \dots$$




- Moreover, analytical treatment during the initial linear regime is easier than in other models.

# Inflaton oscillations in monomial potentials

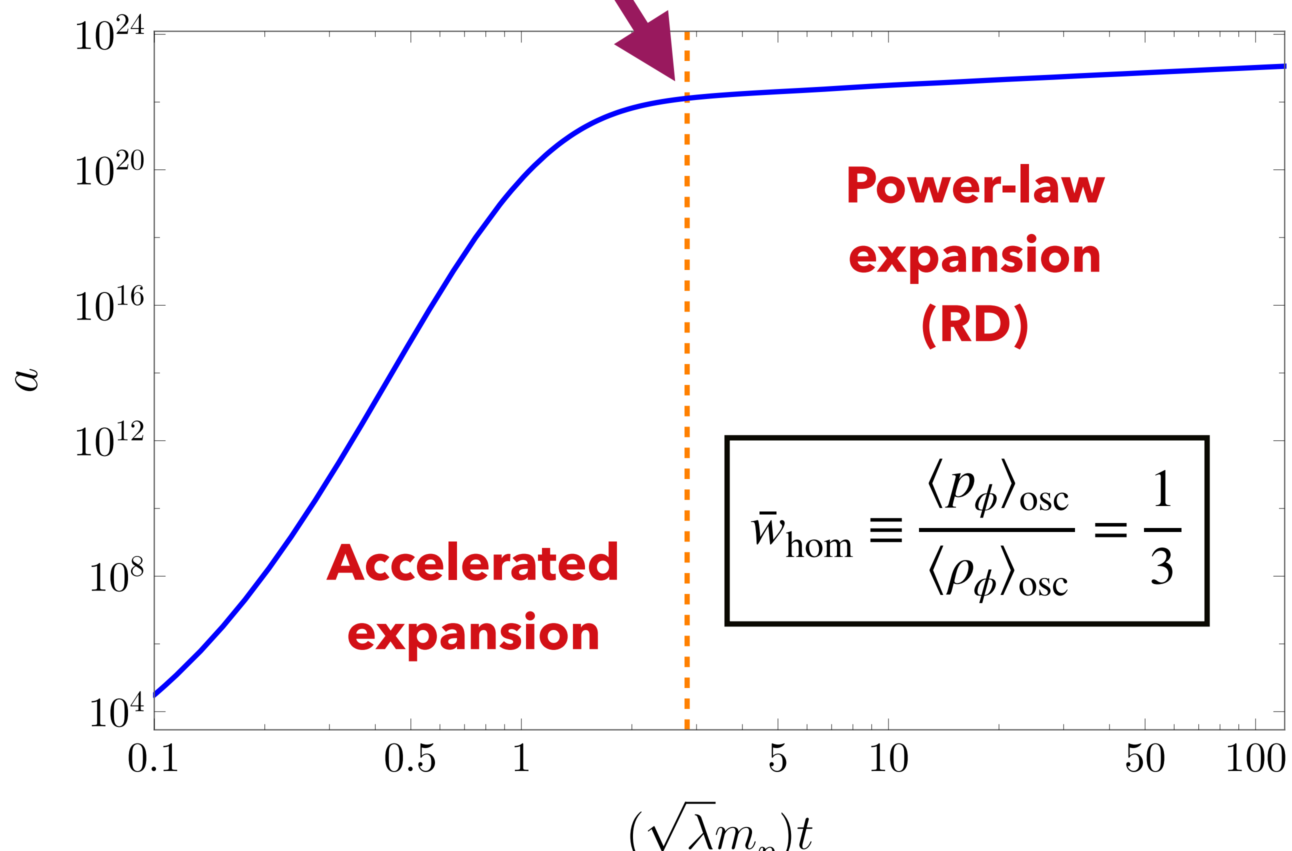



$$\boxed{V(\phi) = \frac{1}{4}\lambda\phi^4}$$

Homogeneous solution:

$$\phi = \bar{\phi}(t)$$

**END OF INFLATION**
$(\phi = \phi_*)$




$$\boxed{\bar{w}_{\rm hom} \equiv \frac{\langle p_\phi\rangle_{\rm osc}}{\langle \rho_\phi\rangle_{\rm osc}} = \frac{1}{3}}$$

**INFLATON OSCILLATIONS:**

**Amplitude:** $A_\phi(t) \equiv \phi_* \left(\frac{t}{t_*}\right)^{-1/2}$

**Frequency:** $\Omega_{\rm osc} \equiv \lambda^{\frac{1}{2}}\phi_* \left(\frac{t}{t_*}\right)^{-\frac{1}{2}}$

Turner, PRD (1983)

# Inflaton oscillations in monomial potentials

**Generic expressions for monomial potentials:**

$$\boxed{V(\phi) = \frac{1}{p}\lambda\mu^{4-p}|\phi|^p} \qquad p \geq 2$$

**INFLATON OSCILLATIONS:**

$$\phi(t) \simeq \underbrace{A_\phi(t)}_{\text{Decaying amplitude}}\underbrace{F(t)}_{\text{Oscillatory function}}$$

$$\boxed{A_\phi(t) \equiv \phi_* \left(\frac{t}{t_*}\right)^{-2/p}}$$

$$\boxed{\Omega_{\rm osc} \equiv \lambda^{\frac{1}{2}}\mu^{\frac{4-p}{2}}\phi_*^{\frac{p-2}{2}}\left(\frac{t}{t_*}\right)^{\frac{2}{p}-1}}$$

(Note: time-dependent frequency except for p=2)

**EQUATION OF STATE:**

$$\boxed{\bar{w}_{\rm hom} \equiv \frac{\langle p_\phi\rangle_{\rm osc}}{\langle \rho_\phi\rangle_{\rm osc}} = \frac{p-2}{p+2}}$$

Turner, PRD (1983)

# Example: preheating in λφ⁴

➤ We couple the inflaton to a massless "daughter" field.

$$\boxed{V(\phi, X) = \frac{1}{4}\lambda\phi^4 + \frac{1}{2}g^2\phi^2X^2}$$

$\phi$ inflaton

$X$ daughter field

➤ Field equations:

$$\boxed{\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3H\dot{\phi} + \lambda\phi^3 + g^2X^2\phi = 0}$$

$$\boxed{\ddot{X} - \frac{1}{a^2}\nabla^2 X + 3H\dot{X} + g^2X^2\phi = 0}$$

$$\boxed{\frac{\ddot{a}}{a} = \frac{1}{3m_p^2}(-\dot{\phi}^2 - \dot{X}^2 + V(\phi, X))}$$

with initial conditions:

$$\phi(t_*, \vec{x}) = \phi_* + \delta\phi(t_*, \vec{x})$$
$$X(t_*, \vec{x}) = \delta X(t_*, \vec{x})$$

vacuum fluctuations at the onset of simulation

➤ We set the initial conditions at the end of inflation. The fluctuations are given by:

$$\left\langle \delta\phi^2 \right\rangle = \int d\log k \ \Delta_{\delta\phi}(k)$$

$$\Delta_{\delta\phi}(k) \equiv \frac{k^3}{2\pi^2}\mathcal{P}_{\delta\phi}(k)$$

$$\boxed{\mathcal{P}_{\delta\phi}(k) \equiv \frac{1}{2a^2\omega_{k,\phi}} \qquad \omega_{k,\phi} \equiv \sqrt{k^2 + a^2m_\phi^2} \qquad m_\phi^2 \equiv \left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=\bar{\phi}_*}}$$

# Example: preheating in λφ⁴

- The inflaton homogeneous mode goes as:

$$A_\phi(t) \equiv \phi_* \left(\frac{t}{t_*}\right)^{-1/2} \propto a^{-1} \qquad \Omega_{\rm osc} \equiv \omega_* \left(\frac{t}{t_*}\right)^{-\frac{1}{2}} \propto a^{-1} \qquad (\omega_* \equiv \sqrt{\lambda}\phi_*)$$

this allows to define a set of dimensionless **"natural variables"** in which the amplitude and oscillation frequency are constant.

- **"Natural" variables** for quartic potential:

conformal rescaling

**Field amplitudes:** $\varphi \equiv \frac{1}{\phi_*} a\phi \qquad \chi \equiv \frac{1}{\phi_*} aX$

**Time and space:** $u \equiv \omega_* \int_{t_*}^{t} a(t')^{-1} dt' \qquad \vec{y} \equiv \omega_* \vec{x}$

conformal time (up to dimensionful factor)

# Example: preheating in λφ⁴

- **Field equations in natural variables:**

$$\varphi'' - \nabla^2_{\vec{y}}\varphi + \left(|\varphi|^2 + q\chi^2 - \underbrace{\cancel{\frac{a''}{a}}}_{\sim u^{-2}}\right)\varphi = 0 \qquad \chi'' - \nabla^2_{\vec{y}}\chi + \left(q\varphi^2 - \underbrace{\cancel{\frac{a''}{a}}}_{\sim u^{-2}}\right)\chi = 0$$

$' \equiv d/du$

| RESONANCE PARAMETER: $q \equiv \dfrac{g^2}{\lambda}$ |
|---|

→ Depends on interaction strength between both fields

- LINEARIZED ANALYSIS: Expansion up to linear order in field amplitudes:

$$\varphi(\vec{y}, u) = \bar{\varphi}(u) + \delta\varphi(\vec{y}, u) + \dots$$

$$\chi(\vec{y}, u) = \delta\chi(\vec{y}, u) + \dots$$

ZEROTH ORDER: $\bar{\varphi}'' + \bar{\varphi}^3 \simeq 0$ → $\bar{\varphi} \simeq \mathrm{cn}(u, 1/2) \approx \cos(0.85u)$

# Example: preheating in λφ⁴

LINEAR ORDER:
$$\begin{cases} \delta\varphi'' - \nabla^2_{\vec{y}}\delta\varphi + 3\bar{\varphi}^2\delta\varphi \simeq 0 \\ \delta\chi'' - \nabla^2_{\vec{y}}\delta\chi + q\bar{\varphi}^2\delta\chi \simeq 0 \end{cases}$$

$$\delta f(\vec{x}, t) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\,\delta f(\vec{k}, t)e^{-i\vec{k}\vec{x}}$$

- Fluctuations (linear regime)
  - **Inflaton:** $\boxed{\delta\varphi_k'' + \tilde{\omega}^2_{k,\varphi}\delta\varphi_k \simeq 0 \quad \tilde{\omega}_{k,\varphi} \equiv \sqrt{\tilde{k}^2 + 3\bar{\varphi}^2}}$
  - **Daughter field:** $\boxed{\delta\chi_k'' + \tilde{\omega}^2_{k,\chi}\delta\chi_k \simeq 0 \quad \tilde{\omega}_{k,\chi} \equiv \sqrt{\tilde{k}^2 + q\bar{\varphi}^2}}$

$$\tilde{k} \equiv \frac{k}{\omega_*}$$

$$\frac{\tilde{\omega}'_{k,f}}{\tilde{\omega}_{k,f}^{\ 2}} \gg 1 \quad \longrightarrow \quad \boxed{\begin{aligned} |\varphi_{\mathbf{k}}|^2 &\sim e^{2\mu_\kappa t} \\ |\chi_{\mathbf{k}}|^2 &\sim e^{2\nu_\kappa(q)t} \end{aligned}}$$

**Floquet index**

**Re[μ$_k$,ν$_k$]>0: resonant excitation!**

**Field fluctuations get excited through a process of PARAMETRIC RESONANCE after inflation!**

**Kofman, Linde, Starobinsky (1994,1997)**

# Example: preheating in λφ⁴

➤ **Stability/instability charts:**

- **For daughter field:**

$$\delta\chi_k'' + (\tilde{k}^2 + q\bar{\varphi}^2)\delta\chi_k \simeq 0$$

$$\downarrow$$

$$|\chi_{\mathbf{k}}|^2 \sim e^{2\nu_\kappa(q)u}$$

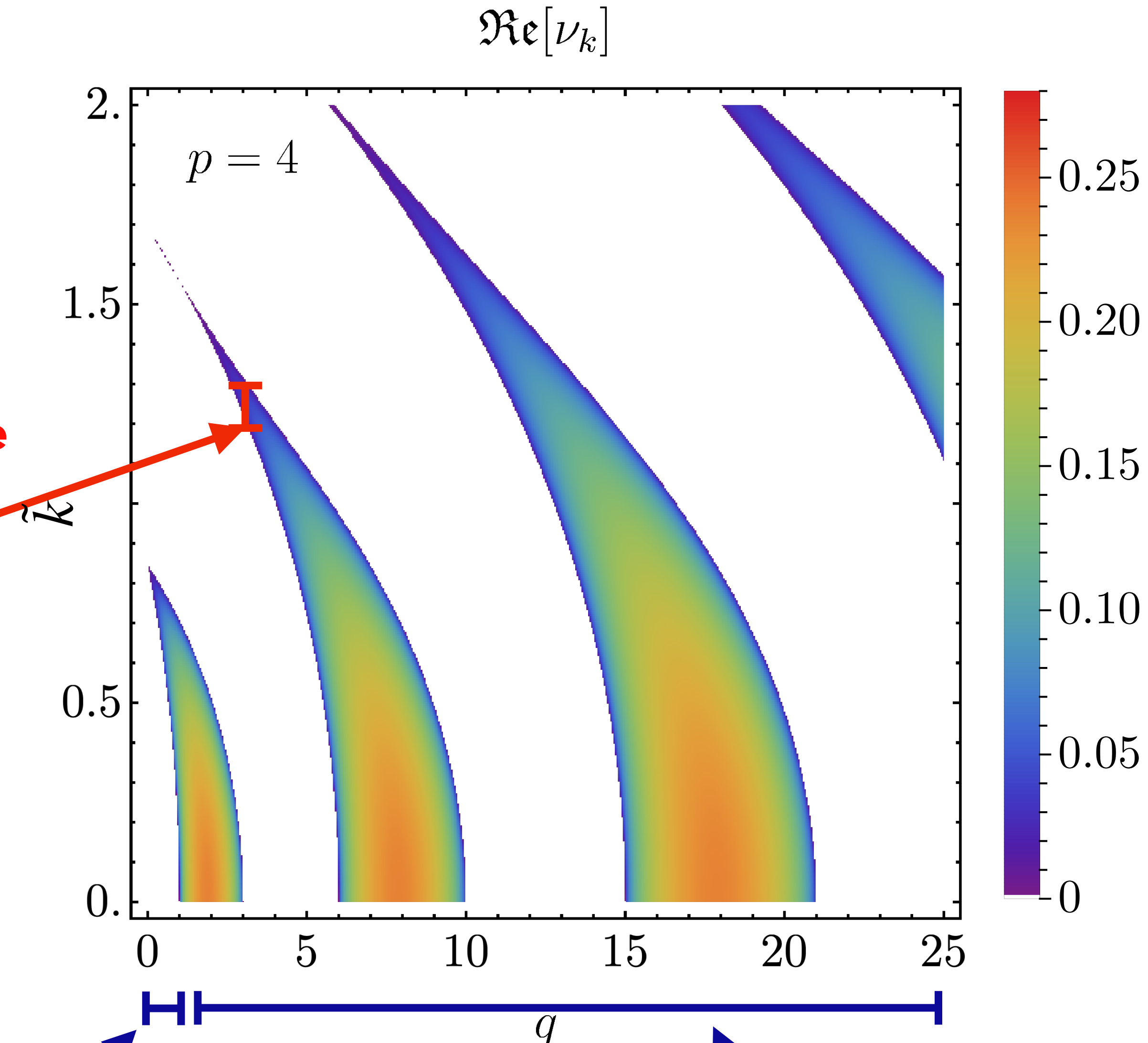
- **For inflaton:**

$$\delta\varphi_k'' + (\tilde{k}^2 + 3\bar{\varphi}^2)\delta\varphi_k \simeq 0$$

$$\downarrow$$

$$|\varphi_{\mathbf{k}}|^2 \sim e^{2\mu_\kappa u}$$

**Inflaton "self"-resonance (q=3)**



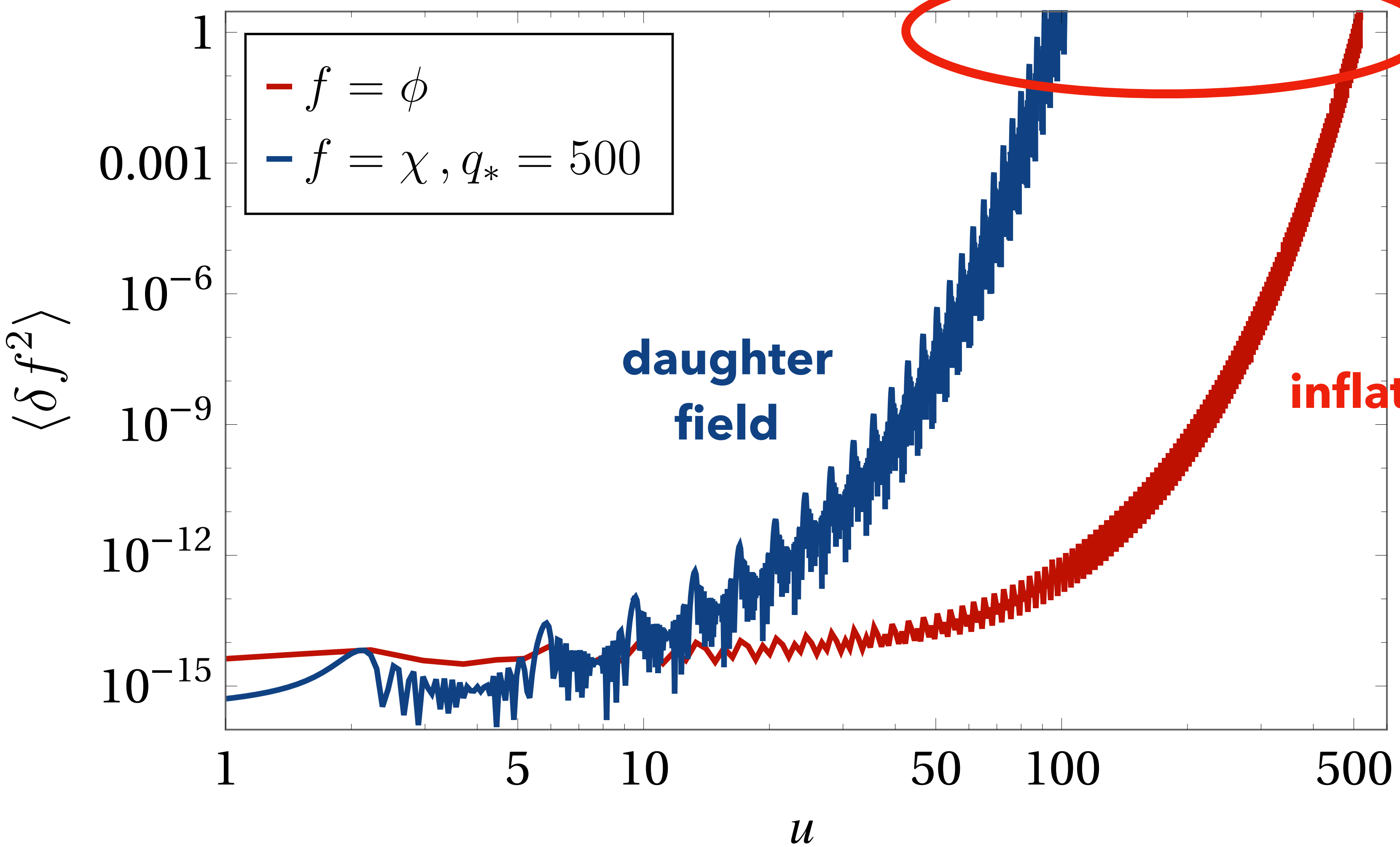

**Narrow resonance (q<1)**

**Broad resonance (q≥1)**

# Example: preheating in λφ⁴

➤ **Variances:** $\langle (\delta f)^2 \rangle \equiv \frac{1}{(2\pi^2)} \int dk k^2 |\delta f_k|^2$

The fluctuations of the daughter field grow much faster than for the inflaton




**SOLUTION NOT VALID IN THIS REGIME (non-linearities become relevant)**

One needs to solve the complete e.o.m in (3+1)-D! (with lattice simulations)

Figure created by Ken Marschall

Thank you!