Exercises for Theoretical Cosmology

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Example exercises

Example 5.1: Comoving curvature perturbation

The *comoving curvature perturbation* $\mathcal{R}$ is given by:

$$\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\bar{\rho} + \bar{P}} q \,, \tag{1}$$

where $T^0_j \equiv -\partial_j q$.

(a) Show that on super-Hubble scales $k \ll \mathcal{H}$, the comoving curvature perturbation $\mathcal{R}$ is constant. Note, on super-Hubble scales we have $\mathcal{H}q = \delta\rho/3$, gradients can be neglected and we are considering adiabatic perturbations.

(b) Show that the amplitude of super-Hubble modes of Φ depends in the following way on the equation of state parameter w:

$$\mathcal{R} \overset{k \ll \mathcal{H}}{\longrightarrow} -\frac{5+3w}{3+3w} \Phi \tag{2}$$

Hint: On scales $k \ll \mathcal{H}$, we have $\Phi' \simeq 0$.

Homework exercises

Exercise 5.1: Two-point correlation function

The two-point correlation function $\xi_{\mathcal{R}}(\boldsymbol{x}, \boldsymbol{x}')$ and the power-spectrum $\Delta^2_{\mathcal{R}}(k)$ are defined as

$$\begin{aligned} \left\langle \mathcal{R}(\boldsymbol{x}) \mathcal{R}\left(\boldsymbol{x}'\right) \right\rangle &\equiv \xi_{\mathcal{R}}\left(\boldsymbol{x}, \boldsymbol{x}'\right) = \xi_{\mathcal{R}}\left(\left|\boldsymbol{x}' - \boldsymbol{x}\right|\right) \,, && (3) \\ \left\langle \mathcal{R}(\boldsymbol{k}) \mathcal{R}^*\left(\boldsymbol{k}'\right) \right\rangle &\equiv \frac{2\pi^2}{k^3} \Delta^2_{\mathcal{R}}(k) \delta_D\left(\boldsymbol{k} - \boldsymbol{k}'\right) \,, && (4) \end{aligned}$$

where $\mathcal{R}(\boldsymbol{x})$ (with $\boldsymbol{x} = \vec{x}$) is the comoving curvature perturbation and $\mathcal{R}(\mathbf{k})$ is its Fourier transform. Show that

$$\xi_{\mathcal{R}}\left(\boldsymbol{x}, \boldsymbol{x}'\right) = \int \frac{\mathrm{d}k}{k} \Delta^2_{\mathcal{R}}(k) \operatorname{sinc}\left(k\left|\boldsymbol{x} - \boldsymbol{x}'\right|\right) , \tag{5}$$

where $\operatorname{sinc}(x) \equiv \sin(x)/x$.

Hint: In case you don't remember, the zeroth order spherical Bessel function can be written as

$$j_0(z) = \frac{1}{2} \int_0^\pi d\theta e^{iz\cos\theta} \sin\theta = \operatorname{sinc}(z) \tag{6}$$

Exercise 5.2: Lyth Bound

In slow-roll inflation, one usually finds that small-field models ($\Delta\phi \ll m_{\mathrm{Pl}}$) have very small tensor-to-scalar ratios r, while they might be observable for large-field models ($\Delta\phi > m_{\mathrm{Pl}}$).

(a) Starting from the relation:

$$N_* = \frac{1}{m_{\text{pl}}} \int_{\phi_e}^{\phi_*} \frac{1}{\sqrt{2\varepsilon(\phi)}} d\phi \tag{7}$$

and assume $\varepsilon \simeq$ const. during inflation, show that the so called *Lyth bound* relates the tensor-to-scalar ratio to the travelled distance $\Delta\phi = \phi_* - \phi_e$ of the inflaton:

$$r = \frac{8}{N_*^2} \left(\frac{\Delta\phi}{m_{\text{pl}}}\right)^2 . \tag{8}$$

(b) With the tensor-to-scalar ratio r and the observed value of the scalar amplitude $A_s = 2.2 \times 10^{-9}$ the energy scale of inflation can be expressed by:

$$V^{1/4} \approx \left(\frac{3}{2}\pi^2 r A_s m_{\text{pl}}^4\right)^{1/4} . \tag{9}$$

<u>Exercise 5.3:</u> Hilltop inflation

In the lecture we have encountered the prototype model of *large-field* inflation (i.e. inflation is happening at large field values $\phi > m_{\text{pl}}$), referred to as *chaotic inflation.* In the following exercise we'd like to consider a specific type of *small-field* inflation (i.e. inflation is happening at small field values $\phi \ll m_{\text{pl}}$), referred to as *hilltop inflation.* In this scenario the scalar potential of the inflaton $\phi(t)$ is given by:

$$V_{\text{inf}}(\phi) = V_0 \left(1 - \frac{\phi^4}{v^4}\right)^2 . \tag{10}$$

(a) Show that slow-roll inflation is possible on the plateau of the potential (i.e. around $\phi \simeq 0$). *Hint:* Show first that the potential can be approximated by $V(\phi) \simeq V_0(1 - 2\phi^4/v^4)$ during inflation.

(b) Calculate the value of the inflaton at the end of inflation ϕ_e, which is given by $|\eta(\phi_e)| \equiv 1$ for hilltop inflation (note that this might depend on the specific model). Also, evaluate ϕ_* via N_*.

(c) With ϕ_* we are now able to evaluate the spectral index $n_s = 1 - 6\varepsilon(\phi_*) + 2\eta(\phi_*)$ and the tensor to scalar ratio $r = 16\varepsilon(\phi_*)$. Compare it to current bounds: $n_s = 0.9665 \pm 0.0038$ and $r < 0.063$. Are models with $N_* = 50 - 60$ in agreement with observations?

(d) Finally we can determine the value of V_0 via the measured value of the scalar amplitude $A_s \simeq 2.2 \times 10^{-9}$.