Exercises for Theoretical Cosmology

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Example exercises

Example 2.1: Chemical potential for electrons

(a) Show that the difference between the number densities of electrons and positrons in the relativistic limit ($m_e \ll T$) is

$$n_e - \bar{n}_e \approx \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right] , \tag{1}$$

where μ_e is the chemical potential. *Hint:* You may need to use

$$\int_0^\infty dy \frac{y}{e^y+1} = \frac{\pi^2}{12} \qquad \text{and} \qquad \int_0^\infty dy \frac{y^2}{e^y+1} = \frac{3\zeta(3)}{2}. \tag{2}$$

(b) Electrical neutrality of the universe implies that the number of protons n_p is equal to $n_e - \bar{n}_e$. Use the baryon to photon ratio $n_\mathrm{B}/n_\gamma \approx 6.1 \times 10^{-9}$ to estimate μ_e/T.

Homework exercises

Exercise 2.1: Pressure and chemical potential

(a) Explain where the $p^2/3E$ term in the expression for pressure is coming from.

(b) Explain what the chemical potential is.

Hint: Have a look into the additional material on ADAM.

Exercise 2.2: Relativistic and non-relativistic limit

Recall the derivation for the number density in the relativistic and non-relativistic limit

(a) Show that in the relativistic limit $P = \rho/3$.

(b) Show that in the non-relativistic limit the energy density is roughly $\rho \simeq mn$ and more accurately given by $\rho = mn + 3nT/2$ (use $E(p) \simeq m + p^2/2m$).

(c) Show that in the non-relativistic limit $P = nT$ and that the equation of state parameter is $w \simeq 0$. Convince yourself that the non-relativistic expression for pressure is the ideal gas law.

(d) Convince yourself that in the non-relativistic limit (i.e. $m \gg T$) the excess of particles over anti-particles is

$$n - \bar{n} = 2g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \sinh\left(\frac{\mu}{T}\right). \tag{3}$$

(e) *Bonus Point*: Show that in the relativistic limit

$$\rho + \bar{\rho} = \frac{7g\pi^2}{120} T^4 \left(1 + \frac{30}{7\pi^2}\left(\frac{\mu}{T}\right)^2 + \frac{15}{7\pi^4}\left(\frac{\mu}{T}\right)^4\right). \tag{4}$$

Exercise 2.3: Number of CMB photons and the neutrino background

(a) Using that the temperature of the CMB is $T_{\text{CMB}} = 2.73K$, show that the number density of CMB photons today is $n_{\gamma,0} \approx 410\,\text{photons/cm}^3$, and compute $\Omega_\gamma h^2 \approx 2.5 \times 10^{-5}$.

(b) Compute the number density of neutrinos and give an estimate of the density parameter $\Omega_\nu h^2$. Recall that the temperature of the neutrino background is $T_0^\nu = 1.95\,\text{K}$ or $0.17\,\text{meV}$ and that they have a non-zero mass m_ν.

Exercise 2.4: Baryon symmetric universe

Consider a universe with an equal amount of baryons b and anti-baryons $\bar{b}$. They become non-relativistic at a temperature $T \sim 1\,\text{GeV}$. Baryons and anti-baryons then start to annihilate. This process keeps efficient until $H \sim \Gamma_b$, where $\Gamma_b = n_b\langle\sigma v\rangle$ is the reaction rate between b and $\bar{b}$ and H the Hubble rate. Give a rough estimate for the freeze-out temperature and the final ratio of the baryon to photon number density. Explain why this value stays constant and compare it to todays observed ratio of $n_B/n_\gamma \sim 6 \times 10^{-10}$.

Use: For the mass of the baryons you can take $m_b \approx 1\,\text{GeV}$ and $\langle\sigma v\rangle \simeq m_\pi^{-2}$ with $m_\pi \approx 135\,\text{MeV}$ to estimate the reaction rate. The degrees of freedom are $g_b = 4$ and $g_* = 10.75$ at the time of freeze out.