Exercises for Theoretical Cosmology

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Example exercises

Example 1.1: Friedmann got hit by an apple

Consider a sphere with mass M and radius $R(t)$ that varies with time.

(a) Derive the continuity equation by assuming that the mass inside the sphere is preserved.

(b) Derive the second and first Friedmann equations from:

$$\ddot{R}(t) = -\frac{GM}{R^2(t)} \, , \tag{1}$$

where G is Newton's constant. Recall the results obtained in the lecture, compare and discuss the differences.

Example 1.2: Ω_M-Ω_Λ parameter space

Describe the regions in the plane $(\Omega_\mathrm{M}, \Omega_\Lambda)$ that correspond to a universe that is: 1) open/closed, 2) expanding/collapsing, and 3) accelerating/decelerating.

Example 1.3: Accelerated Universe

Consider a flat universe filled with dust and a non-zero cosmological constant, such that $\Omega_\Lambda^0 + \Omega_M^0 = 1$. This scenario models the present universe fairly well. The observed values of the density parameters are $\Omega_\Lambda^0 \simeq 0.69$, $\Omega_m^0 \simeq 0.31$, $\Omega_r^0 \simeq 9.4 \times 10^{-5}$ and $|\Omega_k^0| \leq 0.01$.

(a) Show that the scale factor can be expressed as

$$a(t) = \left(\frac{\Omega_m^0}{1-\Omega_m^0}\right)^{1/3} \sinh\left(\frac{3}{2} H_0 \sqrt{1-\Omega_m^0}\, t\right)^{2/3} \, , \tag{2}$$

where we have set the scale factor of today to $a_0 \equiv 1$.

(b) Use the observed value of the Hubble parameter $H_0 = 67.66 \pm 0.42\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ to compute the age of the universe.

Homework exercises

Exercise 1.1: Horizons and distances

Answer the following questions:

(a) Indicate if the *particle horizon* and the *event horizon* in a flat FLRW metric are finite or infinite, when the Universe is filled only by 1) radiation, 2) matter, 3) a cosmological constant.

(b) When measuring distances in cosmology, what is the advantage of using the *luminosity distance* or the *angular distance*, instead of the *metric distance*?

Exercise 1.2: Cosmic time-redshift relation

(a) Derive the relation

$$t_2 - t_1 = \int_{z_2}^{z_1} \frac{dz}{(1+z)H(z)} \,, \tag{3}$$

which relates the cosmic time interval $t_2 - t_1$ and the corresponding redshifts z_1, z_2.

(b) Consider a flat universe containing only matter. By using the relation derived in part (a), show that the age t_0 of such a universe is given by:

$$t_0 = \frac{2}{3H_0} \,. \tag{4}$$

(c) Calculate explicitly the age of this universe (by using $H_0 \simeq 67\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$), and compare it to the result obtained in Example 1.3.

Exercise 1.3: Conservation equation from the Friedmann equations

Consider the Friedmann equations for a spatially-flat FLRW Universe,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \,. \tag{5}$$

Show that these equations necessarily imply the following conservation constraint,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \,. \tag{6}$$

Exercise 1.4: FLRW metric and the Einstein equations

Write a short *Mathematica* notebook in which you evaluate explicitly the Ricci Tensor $R_{\mu\nu}$ and the Ricci scalar R for a FLRW-universe. Using this as a basis, you can derive the Friedmann equations.

Exercise 1.5: Static Universe

Consider a universe with a non-zero cosmological constant Λ and a matter density ρ_m. Show that there is a non-flat solution for which a is constant, and compute the required value of Λ.

Exercise 1.6: Friedmann equations in conformal time

(a) Using the Friedmann equations in cosmic time t, derive the Friedmann equations in conformal time $d\eta \equiv dt/a(t)$,

$$\left(a'\right)^2 + ka^2 = \frac{8\pi G}{3}\rho a^4 \ , \tag{7}$$

$$a'' + ka = \frac{4\pi G}{3}(\rho - 3p)a^3 \ , \tag{8}$$

where $' \equiv d/d\eta$ and $\mathcal{H} \equiv a'/a$.

(b) Show that for a universe dominated by a fluid with equation of state $\omega \equiv p/\rho$, the scale factor goes as $a(\eta) \sim \eta^{2/(1+3w)}$. How does the Hubble parameter $\mathcal{H}$ evolve? Particularise the solutions for a RD and MD universe.

(c) How does the scale factor $a(\eta)$ and the Hubble parameter $\mathcal{H}(\eta)$ evolve in a universe dominated by dark energy ($\omega = w_\Lambda \equiv -1$)?