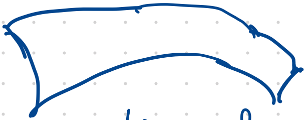
① Geometry & dynamics of the universe

1.1. Summary on GR
1.2. FLRW metric
1.3. Matter sources
1.4. Friedmann equations
1.5. Horizons

1.1. Summary on GR



- spacetime coordinates: $X^\mu = (t, x^i)$

$\mu = 0,1,2,3 \quad i = 1,2,3 \quad \underline{X}$

- Line element: $ds^2 = \sum_{\mu,\nu} \underset{\uparrow \text{ spacetime metric}}{g_{\mu\nu}} d\underline{X}^\mu d\underline{X}^\nu = g_{\mu\nu} d\underline{X}^\mu d\underline{X}^\nu$

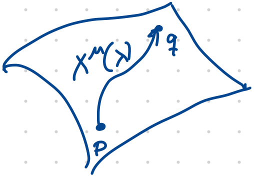
$$ds^2 = \underset{\uparrow \text{speed of light}}{c^2} d\tau^2 \qquad \tau : \boxed{\text{proper time}}$$

$$c = 1 \Rightarrow ds^2 = d\tau^2$$

- Minkowski spacetime: $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ [spec. rel.]
- Curved spacetime: $g_{\mu\nu} = g_{\mu\nu}(t, \vec{x})$ [gen. rel]

KINEMATICS

$X^\mu(\lambda)$ λ: parameter



$$ds^2 \begin{cases} > 0 \rightarrow \text{timelike curve} \rightarrow \text{massive particles} \\ = 0 \rightarrow \text{null curve} \rightarrow \text{massless particles (ex. photons)} \\ < 0 \rightarrow \text{spacelike curve} \rightarrow \text{negative mass?} \end{cases}$$

In absence of non-gravitational forces, particles move along **geodesics**.

- Massive particles: curves that maximize proper time.

$\lambda = \tau$ $$\tau = \int_{\tau_p}^{\tau_q} d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \xRightarrow{\delta\tau = 0}$$ Euler-Lagrange eqs

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \underbrace{\Gamma^\mu{}_{\alpha\beta}} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0}$$ Geodesic equation

Christoffel symbols: $$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} \left(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \right)$$

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$$

$U^\mu = \dfrac{dx^\mu}{d\tau}$: 4-velocity $$\frac{dU^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta} U^\alpha U^\beta = 0 \to \boxed{U^\alpha \nabla_\alpha U^\mu = 0}$$

covariant derivative $$\nabla_\alpha U^\mu \equiv \partial_\alpha U^\mu + \Gamma^\mu{}_{\alpha\beta} U^\beta$$

$$p^\mu = m U^\mu \longrightarrow \boxed{p^\alpha \nabla_\alpha p^\mu = 0}$$

$ds = d\tau$

- Massless particles:

$d\tau = 0 \Rightarrow$ no natural λ $\Rightarrow$ but still $$s = \int_{\lambda_p}^{\lambda_q} d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

$$\xRightarrow{\delta S = 0} p^\mu = \frac{dx^\mu}{d\lambda} \Rightarrow p^\mu = (E, p^i) \Rightarrow \boxed{p^\alpha \nabla_\alpha p^\mu = 0}$$

DYNAMICS

Einstein equations $$\boxed{G_{\mu\nu} = 8\pi T_{\mu\nu}}$$

$$\boxed{g_{\mu\nu}(x)} \to \boxed{\Gamma^\mu{}_{\alpha\beta}} \to \boxed{R^\alpha{}_{\beta\gamma\delta}} \to \boxed{R_{\mu\nu}} \to \boxed{R} \to \boxed{G_{\mu\nu}}$$

metric, Christoffel symbol, Riemann tensor, Ricci tensor, Ricci scalar

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\beta\epsilon} \Gamma^\epsilon{}_{\beta\delta} - \Gamma^\alpha{}_{\delta\epsilon} \Gamma^\epsilon{}_{\beta\sigma}$$

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \qquad R = R^\mu{}_\mu \qquad \boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$$

$T_{\mu\nu} \Rightarrow$ energy-momentum tensor

energy density : ρ

$$T_{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

$T_{\mu\nu} = T_{\nu\mu}$

pressure

$T^{11} \to x$

$T^{22} \to y$

$T^{33} \to z$

momentum density

$T^{10} \to x$

$T^{20} \to y$

$T^{30} \to z$

shear stress

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}$$

$$\nabla_\mu T^{\mu}{}_{\nu} = \partial_\sigma T^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\sigma\alpha} T^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\sigma\nu} T^{\mu}{}_{\alpha} = 0$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Spacetime curvature $\Longleftrightarrow$ Content of the universe

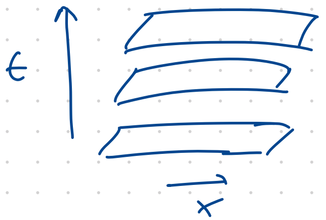
1.2. FLRW metric

1.2.1. Metric form

COSMOLOGICAL PRINCIPLE: Universe is homogeneous and isotropic at large scales

$$\underset{\uparrow\ \text{sec. 1.2}}{G_{\mu\nu}} = 8\pi G \underset{\nwarrow\ \text{sec. 1.3}}{T_{\mu\nu}}$$

ordered time slices



$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - \underbrace{R^2(t)}_{\text{scale factor}}\, d\ell^2$$

$$d\ell^2 = \underbrace{\gamma_{ij}}_{\rightarrow \text{spatial 3D metric}} dx^i dx^j \qquad \gamma_{ij} = f(t, \vec{x})$$

γ_{ij} maximally symmetric $\rightarrow$ constant 3-curvature $\;{}^{(3)}R = 6K$

$$\left[d\ell^2 = \frac{d\bar{r}^2}{1 - K \bar{r}^2} + \bar{r}^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] \quad K \begin{cases} = 0 & \text{flat} \\ +1 \rightarrow & \text{closed} \\ -1 \rightarrow & \text{open} \end{cases}$$

(Baumann lectures ch.1)

Comments

① FLRW metric is invariant under rescaling:

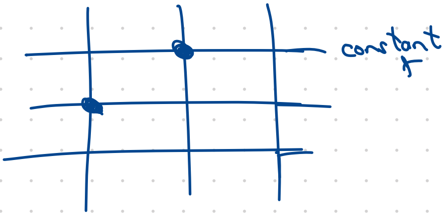
$$ds^2 = dt^2 - \underline{R^2(t)} \left[\frac{d\bar{r}^2}{1 - K\bar{r}^2} + \bar{r}^2 d\Omega^2 \right] \qquad a(t_0) = 1$$

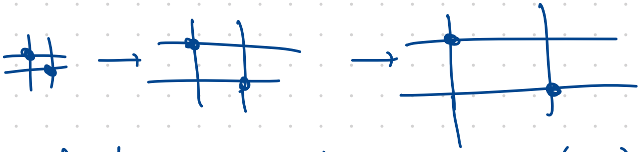
$R(t) = (\text{length})^1$	$\lambda = (\text{length})$	
$\bar{r} = (\text{length})^0$	$R(t) \rightarrow \lambda a(t)$	$a(t) = (\text{length})^0$
$K = \{-1, 0, +1\}^0$	$r \rightarrow \lambda^{-1} r$	$r = (\text{length})^1$
	$K \rightarrow \lambda^2 K$	$K = (\text{length})^{-2}$

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$$\lambda = (\text{length}) = R(\underset{\uparrow\ \text{current time}}{t_0}) \Rightarrow \underline{a(t_0) = 1}$$

② Even if r is constant, particles move away





Physical velocity: $\underline{\underline{v_{ph}}} = \frac{dr_{phys}}{dt} = \frac{d(ar)}{dt} = \underbrace{a(t)\overset{r=0}{\frac{dr}{dt}}}_{\substack{v_{pec}\\ \text{peculiar}}} + \underbrace{\frac{da}{dt}r}_{\substack{\text{Hubble}\\ \text{flow}}}$

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$$

$$= v_{pec} + H r_{phys}$$

If $v_{pec} \ll H r_{phys} \Rightarrow \boxed{v_{ph} \simeq H r_{phys}}$ Hubble's law

$\underline{H_0} = H(t_0) = 100\ h\ \text{km}\ s^{-1}\ \text{Mpc}^{-1} \quad h \simeq 0.67 \pm 0.01$

3) Convenient radial transformation $d\chi = \frac{dr}{\sqrt{1-kr^2}}$

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)\, d\Omega^2\right]$$

$\uparrow$ it simplifies $\underline{\underline{g_{rr}}}$

$$S_k(\chi) = \frac{1}{\sqrt{k}}\begin{cases} \sinh(\sqrt{k}\chi) & k<0 \\ \sqrt{k}\chi & k=0 \\ \sin(\sqrt{k}\chi) & k>0 \end{cases}$$

④ Conformal time $d\eta = \frac{dt}{a(t)} \rightarrow \boxed{ds^2 = a^2(\eta)\left[d\eta^2 - (d\chi^2 + S_k^2(\chi)\, d\Omega^2\right]}$

$\uparrow$

t → cosmic time

1.2 FLRW metric

$$ds^2 = dt^2 - a^2(t)\,\delta_{ij}\,dx^i dx^j = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

$$K = \{-1, +0, +1\}$$

1. 2.2. Particles in an expanding universe

$$p^\alpha \nabla_\alpha p^\mu = 0 \rightarrow p^\alpha\left(\partial_\alpha p^\mu + \Gamma^\mu_{\alpha\beta}\, p^\beta\right) = 0 \quad \leftarrow \underline{\underline{\mu = i}}$$

$$\downarrow \mu = 0$$

$$p^\alpha\left(\partial_\alpha p^0 + \Gamma^0_{\alpha\beta}\, p^\beta\right) = 0$$

$$\downarrow \quad \partial_i p^\mu = 0 \quad \text{(FLRW homogeneity)}$$

$$p^0 \partial_0 p^0 + \Gamma^0_{\alpha\beta}\, p^\alpha p^\beta = 0 \quad \leftarrow \Gamma^0_{0i} = \Gamma^0_{00} = 0$$

$$\downarrow \quad \leftarrow \Gamma^0_{ij} = \dot a a\,\delta_{ij}$$

$$p^0 \partial_0 p^0 + \dot a a\,\delta_{ij}\, p^i p^j = 0$$

$$\boxed{p^0 \dot p^0 = -\frac{\dot a}{a} p^2}$$

$g_{ij} = -a^2\delta_{ij}$ three-momentum

$$\downarrow \qquad \swarrow$$

$$\underline{\underline{g_{ij}}}\, p^i p^j = -p^2$$

$$\underline{a^2 \delta_{ij} p^i p^j} = p^2$$

Ⓔ Massless particles

$$p^0 = E = p \qquad p\dot p = -\frac{\dot a}{a} p^2 \longrightarrow \ln p = -\ln a + \text{const}$$

$$\boxed{p \propto \frac{1}{a}} \qquad \boxed{E \propto \frac{1}{a}}$$

> The energy of a massless particle decreases as the universe expands

$$\lambda = \frac{h}{p} \propto h a \Rightarrow \underset{\substack{\uparrow \\ \text{observed}}}{\lambda_0} = \frac{a(t_0)}{a(t_e)}\underset{\substack{\uparrow \\ \text{emitted}}}{\lambda_e}$$

REDSHIFT

$$\boxed{z = \frac{\lambda_0 - \lambda_e}{\lambda_e}} \rightarrow$$

$$\lambda_0 > \lambda_e \rightarrow z > 0$$

$$a(t_0) = 1 \rightarrow \boxed{1 + z = \frac{1}{a(t)}}$$

$$z(t_0) = 0$$

$$z(t_{CMB}) \simeq 1100$$

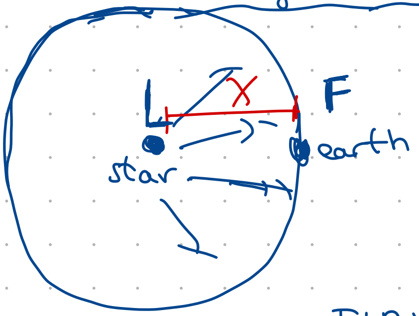
1.2.3 Distances

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S_K^2(\chi)\,d\Omega^2\right] \; ; \; S_K(\chi) = \frac{1}{\sqrt{K}} \begin{cases} \sinh(\sqrt{K}\chi) & \leftarrow K<0 \\ \sqrt{K}\,\chi & K=0 \\ \sin(\sqrt{K}\chi) & \leftarrow K>0 \end{cases}$$

- Comoving distance: χ
- Metric distance: $d_m = S_K(\chi) \overset{K=0}{=} \chi$

NOT observable!

- Luminosity distance:



– Euclidean space

$$F = \frac{L}{4\pi\chi^2}$$

$L \leftarrow$ absolute luminosity (energy/second)

F: $\uparrow$ $\frac{\text{energy}}{\text{second . area}}$

–FLRW spacetime: the universe expands

$$F = \frac{L}{4\pi d_m^2} \times \left(\frac{a(t_e)}{a(t_0)}\right) \times \left(\frac{a(t_e)}{a(t_0)}\right) = \frac{L}{4\pi d_m^2}\,\frac{1}{(1+z)^2} = \frac{L}{4\pi d_L^2}$$

F: $\uparrow$ observed flux

first factor: $\uparrow$ energy redshift

second factor: $\uparrow$ rate of arrival of photons

$$\boxed{d_L = d_m(1+z)}$$

- Angular distance:

$[\delta\theta \ll 1]$

– Euclidean spacetime:

$$\chi = \frac{D}{\delta\theta}$$

–FLRW spacetime:

$$d_A = \frac{D}{\delta\theta} \underset{\uparrow}{=} a(t_e)\,S_K(\chi) = \frac{d_m}{1+z}$$

$$D = a(t_e)\,S_K(\chi)\,\delta\theta$$

$$\boxed{d_A = \frac{d_m}{1+z}}$$

$$\boxed{(1+z)\,d_A = d_m = \frac{d_L}{1+z}}$$

$$ds^2 = 0 \rightarrow \chi(z) = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^z \frac{dz}{H(z)}$$

t_0 ← current time

t_e → emission time

← depends on the content of the universe

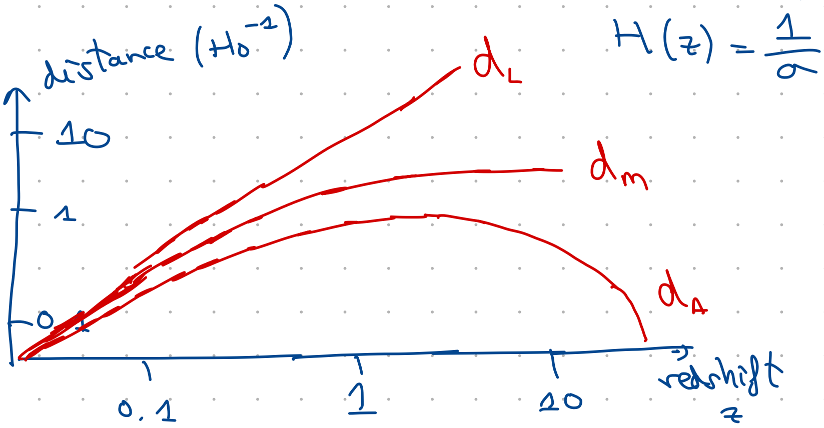
$$1+z = \frac{1}{a}$$

$$H(z) = \frac{1}{a}\frac{da}{dt}$$

present universe

70% DE

30% matter



(1.3) Matter sources

energy density

$$T_{\mu\nu} = \begin{pmatrix} T^{00} & & & \\ T^{10} & T^{11} & & \\ T^{20} & T^{21} & T^{22} & \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

momentum density

pressure

→ shear stress

- Homogeneity + isotropy:

Comoving observer $u^\mu = (1,0,0,0)$

– rest

– no preferred direction in space

$$T^\mu{}_\nu = g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & -P(t) & 0 & 0 \\ 0 & 0 & -P(t) & 0 \\ 0 & 0 & 0 & -P(t) \end{pmatrix} \Rightarrow$$

Generalized observer

$$\boxed{T^\mu{}_\nu = (\rho+P)\,U^\mu U_\nu - P\,\delta^\mu_\nu}$$

Any matter content that is described by this tensor is called a PERFECT FLUID.

- $\nabla_\mu T^{\mu\nu} = 0 \rightarrow \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\mu\lambda} T^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu} T^\mu{}_\lambda = 0$ (4 equations)

$$\nu = 0 \quad \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}$$

conservation equation

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho+P) = 0} \longleftarrow$$

EQUATION OF STATE

$$w = \frac{P}{\rho} = \text{const}$$

$$\frac{d\rho}{dt} + \frac{3}{a}\frac{da}{dt}\underbrace{(\rho+P)}_{\rho(1+w)} = 0 \rightarrow \frac{d\rho}{\cancel{dt}} + \frac{3}{a}\frac{da}{\cancel{dt}}\rho(1+w) = 0 \longrightarrow$$

$$\rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \rightarrow \boxed{\rho \propto a^{-3(1+w)}}$$

1.3.2 COSMIC INVENTORY

- Matter $w=0$

$|P| \ll \rho \rightarrow \rho \propto a^{-3}$

- Baryons: Ordinary matter (nuclei, electrons, some beyond the SM particles)
- Dark matter: ??

- Radiation $w = \frac{1}{3}$

$P = \frac{1}{3}\rho \longrightarrow \rho \propto a^{-4}$

- Photons
- Neutrinos
- Gravitons

- Dark energy $w = -1$

$P = -\rho \longrightarrow \rho \propto a^{0} \equiv \text{constant}$

Energy is created as the universe expands

$E = \rho V \sim a^{3}$

- Vacuum energy: QFT predicts $T^{vac}_{\mu\nu} = \rho_{vac}\, g_{\mu\nu}$ $\quad \frac{\rho_{vac}^{(predicted)}}{\rho_{vac}^{(observed)}} \sim 10^{120}$

- Cosmological constant:

$$G_{\mu\nu} - \underset{\text{cosmological constant}}{\Lambda}\, g_{\mu\nu} = 8\pi G T_{\mu\nu} \rightarrow G_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}\right)$$

$$T^{(\Lambda)}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} = \rho_{\Lambda}\, g_{\mu\nu}$$

- Others (p.x. modified GR)

(1.4) Friedmann equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix}$$

$$ds^2 = dt^2 - a^2(t)\delta_{ij}\,dx^i dx^j$$

- Ricci tensor $R_{\mu\nu}$

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2K}{a^2}\right] g_{ij}$$

- Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right]$$

Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$G^0{}_0 = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right]$$

$$G^i{}_j = \left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right)\delta^i{}_j$$

$\mu\nu = 0,0 \rightarrow$ $$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$$ 1st Friedmann equation

$\mu\nu = i,i \rightarrow$ $$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$ 2nd Friedmann equation

$$\rho = \rho_m + \rho_r + \rho_\Lambda$$

$$P = P_m + P_r + P_\Lambda$$

1.4 Friedmann equations

$$\underset{\text{C.P}}{\underset{\uparrow}{G_{\mu\nu}}} = 8\pi \underset{\text{C.P}}{\underset{\uparrow}{T_{\mu\nu}}} \longrightarrow \boxed{\begin{aligned} \left(\frac{\dot a}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{K}{a^2} \\ \frac{\ddot a}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \end{aligned}}$$

$$\rho = \rho_m + \rho_r + \rho_\Lambda \qquad \boxed{P_a = w_a \rho_a} \xleftarrow{} \text{equation of state} \qquad w_m = 0 \quad w_r = \frac{1}{3} \quad w_\Lambda = -1$$

$$P = P_m + P_r + P_\Lambda \qquad a = m, r, \Lambda$$

- Comments:

- $1\text{st} + 2\text{nd} \Rightarrow$ exercise $\quad \boxed{\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0}$ $\quad \nabla_\mu T^{\mu\nu}$

- $H(t) = \dfrac{\dot a}{a}$ Hubble parameter $\longrightarrow \boxed{H^2 = \dfrac{8\pi G}{3}\rho - \dfrac{K}{a^2}}$

- critical energy density $\quad \rho_{crit}^{(0)} = \dfrac{3H_0^2}{8\pi G} \underset{\uparrow}{=} 1.1 \times 10^{-5} h^2$ protons cm^{-3}

$$H_0 = H(t_0) = 100\, h\, \text{km}\, \text{s}^{-1}\, \text{Mpc}^{-1} \qquad h = 0.67 \pm 0.01$$

- $\Omega_a^{(0)} = \dfrac{\rho_a^{(0)}}{\rho_{crit}^{(0)}}$ $\quad a = r, m\ \Lambda$ $\qquad \Omega_m^{(0)} = \dfrac{\rho_m^{(0)}}{\rho_{crit}^{(0)}} \quad \Omega_r^{(0)} = \dfrac{\rho_r^{(0)}}{\rho_{crit}^{(0)}} \quad \Omega_\Lambda^{(0)} = \dfrac{\rho_\Lambda^{(0)}}{\rho_{crit}^{(0)}}$

- $\boxed{\Omega_K^{(0)} = \dfrac{-K}{a_0^2 H_0^2}}$ $\qquad \boxed{\rho_a = \rho_a^{(0)} a^{-3(1+w_a)}}$

$$\rho_m = \rho_m^{(0)} a^{-3}$$
$$\rho_r = \rho_r^{(0)} a^{-4}$$
$$\rho_\Lambda = \rho_\Lambda^{(0)}$$

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{K}{a^2}$$
$$\downarrow$$
$$H^2 = H_0^2\left[\Omega_r^{(0)}\left(\frac{a_0}{a}\right)^4 + \Omega_m^{(0)}\left(\frac{a_0}{a}\right)^3 + \Omega_K^{(0)}\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda^{(0)}\right]$$

$a = a_0$

$H = H_0$

$$\boxed{\Omega_r^{(0)} + \Omega_m^{(0)} + \Omega_K^{(0)} + \Omega_\Lambda^{(0)} = 1},$$

"COSMIC SUM RULE"

$$z = \frac{a_0}{a} - 1 \rightarrow H^2 = H_0^2\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]$$

1.4.1. Observations (CMB+ LSS)

$$\Omega_r^{(0)}+\Omega_m^{(0)}+\Omega_k^{(0)}+\Omega_\Lambda^{(0)} = 1$$

$$\Omega_\Lambda^{(0)} = 0.68$$

"baryonic"

$$\Omega_m^{(0)} = 0.32 \rightarrow \Omega_b = 0.05$$

$$\rightarrow \Omega_c = 0.27$$

cold dark matter

$$\Omega_r^{(0)} \simeq 9\times10^{-5}$$

$$|\Omega_k^{(0)}| \leq 0.01$$

- Universe contains dark energy.
- Universe is (almost) spatially flat.

$$H^2 = H_0\left(\Omega_r^{(0)}\left(\frac{a_0}{a}\right)^4 + \Omega_m^{(0)}\left(\frac{a_0}{a}\right)^3 + \Omega_k^{(0)}\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda^{(0)}\right)$$

$a\to0;\ \frac{a_0}{a}\to\infty$

RD → MD → ΛD

RD: radiation domination, early times

MD: matter domination

ΛD: cosmological constant domination, late times

1.4.2. Solutions

- Single component universe

$$w = \frac{P}{\rho} \Rightarrow \rho \sim a^{-3(1+w)} \qquad \begin{bmatrix} w=0 & MD \\ w=1/3 & RD \\ w=-1 & \Lambda D \end{bmatrix}$$

$$\left(\frac{\dot a}{a}\right)^2 \propto \rho \rightarrow \left(\frac{da}{dt}\right)^2 \propto a^{-(1+3w)} \rightarrow da\, a^{\frac{1+3w}{2}} \propto dt$$

- $w \neq -1 \rightarrow t \propto a^{\frac{3+3w}{2}} \Rightarrow \boxed{a(t) \propto t^{\frac{2}{3(1+w)}}}$
  - ↳ $w=0$ (MD) $\rightarrow a(t) \sim t^{2/3}$
  - ↳ $w=\frac{1}{3}$ (RD) $\rightarrow a(t) \sim t^{1/2}$
- $w=-1$: [ΛD] $\rightarrow \frac{da}{a} \propto dt \rightarrow a(t) = e^{Ht}$

H is a constant

- Matter-radiation equality

$$H^2 = H_0^2\left[\Omega_r^{(0)}\left(\frac{a_0}{a}\right)^4 + \Omega_m^{(0)}\left(\frac{a_0}{a}\right)^3\right]$$

$\rho_m = \rho_r$

$$\Omega_r^{(0)}\left(\frac{a_0}{a}\right)^4 = \Omega_m^{(0)}\left(\frac{a_0}{a}\right)^3 \rightarrow \frac{a_{eq}}{a_0} = \frac{\Omega_R}{\Omega_m} \simeq 3\cdot 10^{-4}$$

$$\downarrow$$

$$\underline{z_{eq} \simeq 3400}$$

- Conformal time η $\quad ds^2 = a^2(d\eta^2 - d\vec{x}^2)$

$$\boxed{dt = a\,d\eta}$$

exercise 1.6

$$\boxed{\begin{aligned} a'^2 + Ka^2 &= \frac{8\pi G}{3}\rho a^4 \\ a'' + Ka &= \frac{4\pi G}{3}(\rho - 3P)a^3 \end{aligned}}$$

$$' = \frac{d}{d\eta}$$

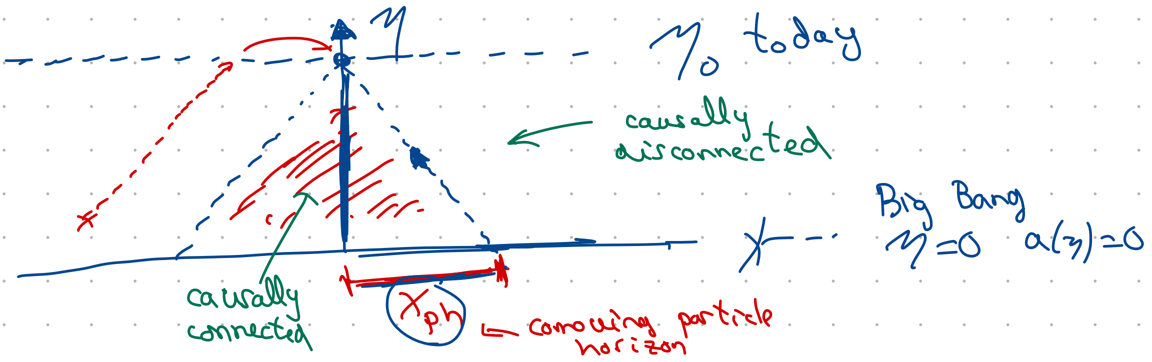
$$a(\eta) \propto \begin{cases} w \neq -1 \rightarrow a(\eta) \sim \eta^{\frac{2}{1+3w}} \begin{cases} \eta^2 & \text{MD} \quad w=0 \\ \eta & \text{RD} \quad w=1/3 \end{cases} \\ \underset{\Lambda D}{w = -1} \rightarrow a(\eta) \sim -\frac{1}{\eta} \end{cases}$$

$$-\infty < \eta < 0$$

1.5 Horizons

- **Particle horizon**: Maximum distance from which light could have travelled to the observer in the present universe.

$$ds^2 = a^2(\eta)\left(d\eta^2 - dx^2\right) \rightarrow \text{photon: } ds^2 = 0 \rightarrow d\eta = dx$$



physical particle horizon:

$$d\eta = a\,dt$$

$$d_{ph} = a(\eta_0)\,\chi_{ph} = a(\eta_0)\,\eta_0 = a(t_0)\int_0^{t_0}\frac{dt'}{a(t')}$$

⊛ RD: $a(t) \sim t^{1/2} \rightarrow d_{ph} = 2t_0 = \frac{1}{H_0} < \infty$

⊛ MD: $a(t) \sim t^{2/3} \rightarrow d_{ph} = 3t_0 = \frac{2}{H_0} < \infty$

⊛ ΛD: $a(t) \sim e^{Ht} \rightarrow d_{ph} = a(t_0)\int_{-\infty}^{t_0} e^{-Ht'}\,dt' = \frac{-a(t_0)}{H}\left[e^{-Ht_0} - \overset{\infty}{e^{+H\infty}}\right] = \infty$

Size of the observable universe

$\Omega_m + \Omega_\Lambda = 1 \Rightarrow$ example 1.3

$$a(t) = \left(\frac{\Omega_m}{1-\Omega_m}\right)^{1/3} \sinh\left(\frac{3}{2}H_0\sqrt{1-\Omega_m}\,t\right)^{2/3}$$

↳ $a(t_0) = 1 \rightarrow t_0 = 13.8\ \text{Gyr}$

$$d_{ph}(t_0) = \underset{1}{\underset{\shortparallel}{a(t_0)}}\int_0^{t_0}\frac{dt'}{a(t')} \underset{\uparrow}{=} 47.1\ \text{Gyr} \underset{c=1}{\underset{\uparrow}{=}} 47.1\ \text{Gly (giga light years)}$$

$\Omega_m = 0.31$

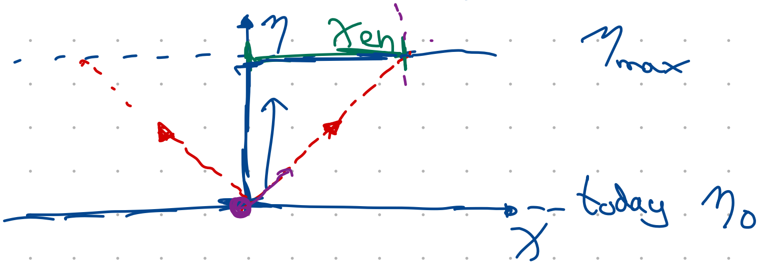
$H_0 = 67.66\,\frac{\text{km}}{\text{s Mpc}} = 0.0691\ \text{Gyr}$

$$\boxed{\text{The expansion of the universe increase the size of our observable universe by a factor} \sim 3.5}$$

Farthest galaxy ever observed: GN-z11 $\quad d_{ph} = 32\ \text{Gly}$.

$\Delta t = t_0 - t = 13.39\ \text{Gyr}$.

Event horizon: Maximum distance that light will be able to reach in the future.



$$d_{eh}(t) = a(t)\chi_{eh} = a(t)(\eta_{max}-\eta) \overset{dt=a\,d\eta}{=} a(t)\int_t^{t_{max}} \frac{dt'}{a(t')}$$

- RD $\Rightarrow a(t) \sim t^{1/2} \rightarrow d_{eh}(t) = t^{1/2}\int_t^{\boxed{t_{max}}\to\infty} dt'\, t'^{-1/2} = +\infty$
- MD $: a(t) \sim t^{2/3} \Rightarrow d_{eh}(t) = \infty$
- ΛD: $a(t) \sim e^{Ht} \Rightarrow d_{eh}(t) = e^{Ht}\int_t^{t_{max}=\infty} dt'\, e^{-Ht'} = \frac{-e^{Ht}}{H}\left[\cancel{e^{-H\infty}} - e^{-Ht}\right]$

$$= \frac{1}{H} < \infty$$

② THERMAL HISTORY OF THE UNIVERSE

2.1. Equilibrium

2.2. Evolution beyond equilibrium $\Rightarrow$ Dark matter freeze-out / Recombination / Nucleosynthesis

2.1.1. Thermal equilibrium

- Rate of interactions: Γ
- Rate of expansion: H

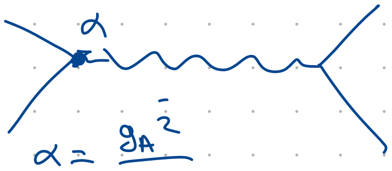
$$\boxed{\Gamma >> H}$$

$$t_c = \frac{1}{\Gamma} << \frac{1}{H} \equiv t_H$$

species are in local thermal equilibrium

Are SM species in thermal equilibrium?

$T > m_{top} \sim 10^2\,\text{GeV}$ → T is only dimensionful scale



$\alpha = \frac{g_A^2}{4\pi}$

$\Rightarrow \Gamma = n\,\sigma\cdot v$ { n: number density; σ: interaction cross-section; v: average velocity

- $v \simeq 1$ $(=c)$ → ultrarelativistic limit
- $n \sim T^3$
- $\sigma \sim \frac{\alpha^2}{T^2}$

$$\Gamma = n\sigma v \sim T^3\frac{\alpha^2}{T^2} = \alpha^2 T$$

$\alpha \simeq 0.01$

$$M_{pl} = \frac{1}{\sqrt{8\pi G}} \qquad H = \frac{\sqrt{\rho}}{M_{pl}} \sim \frac{T^2}{M_{pl}}$$

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 T}{T^2/M_{pl}} \lesssim \frac{10^{16}\,\text{GeV}}{T}$$

Yes: Thermal equilibrium between SM particles at $10^2\,\text{GeV} < T < 10^{16}\,\text{GeV}$.

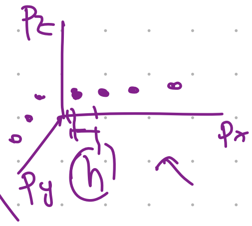
2.1.2. Distributions in thermal equilibrium

(for gas of weakly interacting particles)

number density $\quad n = \frac{g}{(2\pi)^3}\int d^3\vec{p}\, f(p)$ $\qquad f(p)$: distribution function

g: internal degrees of freedom (e.g. spin)

f(p): distribution function. Isotropy: $f(\vec{p}) = f(p = |\vec{p}|)$



$$\text{density of states} \sim \frac{1}{h^3}\times g \overset{\hbar = \frac{h}{2\pi} = 1}{=} \frac{g}{(2\pi)^3}$$

number density: $n = \frac{g}{(2\pi)^3} \int d^3\vec{p}\, f(p)$

energy density: $\rho = \frac{g}{(2\pi)^3} \int d^3\vec{p}\, f(p)\, E(p)$ $\quad E(p) = \sqrt{p^2+m^2}$

pressure density: $P = \frac{g}{(2\pi)^3} \int d^3\vec{p}\, f(p)\, \frac{p^2}{3E(p)}$ ← exercise 2.1

- Thermal equilibrium: Kinetic + chemical equilibrium

- Kinetic equil: Particles exchange energy & momentum efficiently

μ: chemical potential

$$\boxed{f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}} \left\{ \begin{array}{l} +: \text{fermions} \rightarrow \text{Fermi-Dirac distrib.} \\ -: \text{bosons} \rightarrow \text{Bose-Einst. distrib.} \end{array} \right.$$

↳ $T \ll (E-\mu) \Rightarrow \boxed{f(p) \simeq e^{-(E(p)-\mu)/T}}$

Maxwell-Boltzmann distr.

- chemical equilibrium: Rates of forward and reverse reaction are equal

$1+2 \longleftrightarrow 3+4 \quad \Longrightarrow \quad \mu_1 + \mu_2 = \mu_3 + \mu_4$

↳ $\mu_\gamma = 0$. Number of photons is not conserved

Compton scattering: $e^- + \gamma \longleftrightarrow e^- + \gamma + \gamma \quad \mu_\gamma = 0$

↳ $\mu_X = -\mu_{\bar{X}} \qquad X + \bar{X} \longleftrightarrow \gamma + \gamma$

Thermal equilibrium ⇒ species share common temperature T

$T (= T_\gamma)$

⊛ Limits

- At early times: $\mu \to 0$ [example 2.1]

$x = \frac{m}{T}$ $\qquad n = \frac{g}{2\pi^2} T^3 I_\pm(x) \qquad I_\pm(x) = \int_0^\infty d\xi \frac{\xi^2}{e^{\sqrt{\xi^2+x^2}} \pm 1}$

$\xi = \frac{p}{T}$ $\Rightarrow$ $\rho = \frac{g}{2\pi^2} T^4 J_\pm(x) \qquad J_\pm(x) = \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+x^2}}{e^{\sqrt{\xi^2+x^2}} \pm 1}$

- Relativistic limit

$x \to 0 \quad (m \ll T)$

$\zeta(3) \simeq 1.20205$: Riemann zeta-function

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \times \left\{ \begin{array}{ll} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{array} \right.$$

$$\rho = \frac{\pi^2}{30} g T^4 \times \left\{ \begin{array}{ll} 1 & \text{bosons} \\ 7/8 & \text{fermions} \end{array} \right.$$

[exercise 2.2] ↳ $P = \frac{1}{3}\rho \Rightarrow w = \frac{P}{\rho} = \frac{1}{3}$ //

- **Non-relativistic limit**

$T << m$

$x \to \infty$

$$n = g\left(\frac{mT}{2\pi}\right)^{+3/2} e^{-m/T}$$

exercise 2.2.

$$\rho \simeq mn + \frac{3}{2}nT + \dots$$

$$P = nT << \rho \implies w \simeq \frac{P}{\rho}$$

⊛ Effective number of relativistic SM species

$$\rho_r = \sum_i \rho_i = \frac{\pi^2}{30} g_*(T)\, T^4$$

i: sum over all SM species with $T >> m_i$

If all species are in thermal equilibrium, $T_i = T$,

$$g_*^{th}(T) = \underbrace{\sum_{i=b} g_b}_{\text{bosons}} + \frac{7}{8}\underbrace{\sum_{i=f} g_i}_{\text{fermions}}$$

If some species are not in thermal equilibrium $T_i \neq T$

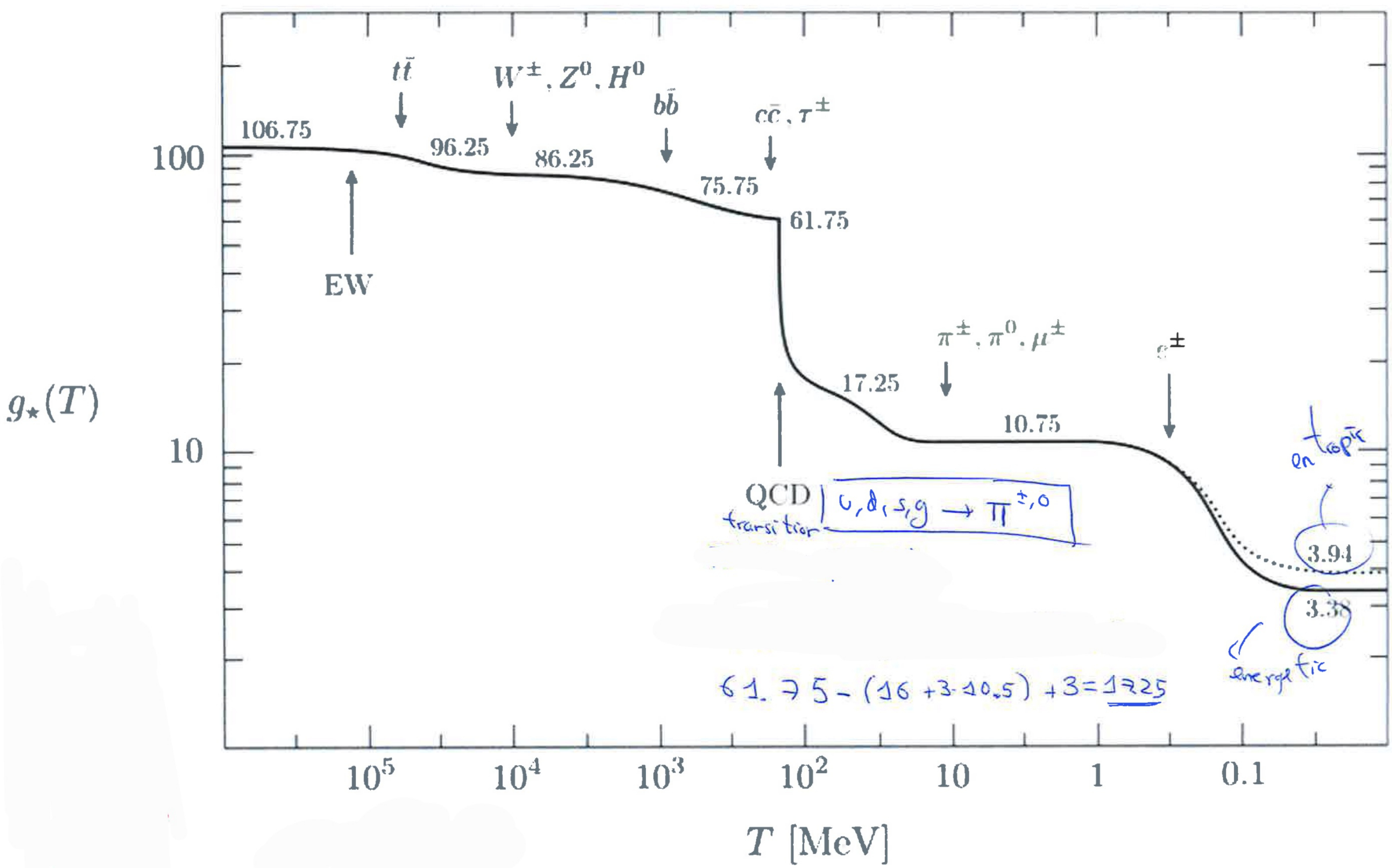
$$g_*^{dec}(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4$$

- For $T >> 10^2\,\text{GeV} \sim m_{top}$

$$g_*^{th}(T) = \Big(\overset{\text{Higgs}}{1} + \underset{\text{spins}=+1,-1}{\overset{\text{photons}}{2}} + \overset{W^+,W^-,Z^0}{\underset{\substack{\text{spins}\\+1,-1,0}}{3}\times 3} + \overset{\text{gluons}}{\underset{\text{spin}}{2}\times\underset{\text{colour}}{8}} \Big)$$

$$+ \frac{7}{8}\Big(\underset{\substack{\text{charged leptons:}\\ e^-,\mu^-,\tau^-}}{3\times\overset{\text{charge}}{2}\times\overset{\text{spin}}{2}} + \underset{\text{quarks}}{6\times\overset{\text{charge}}{2}\times\overset{\text{spin}}{2}\times\overset{\text{colour}}{3}} + \underset{\substack{\text{neutrinos}\\ \nu_e,\nu_\mu,\nu_\tau}}{3\times\overset{\text{spin}}{2}} \Big) =$$

$$= 106.75$$


$g_\star(T)$
100
10
106.75
$t\bar{t}$
$W^\pm, Z^0, H^0$
96.25
86.25
$b\bar{b}$
75.75
$c\bar{c}, \tau^\pm$
61.75
EW
17.25
$\pi^\pm, \pi^0, \mu^\pm$
10.75
$e^\pm$
QCD
transition
$u, d, s, g \rightarrow \pi^{\pm,0}$
entropic
3.94
3.38
energetic
$61.75 - (16 + 3 \cdot 10.5) + 3 = 17.25$
10^5
10^4
10^3
10^2
10
1
0.1
T [MeV]

2.1.3 Conservation of entropy

Proof: entropy is conserved

$$U = \rho \cdot V$$

$$T\,dS = dU + P\,dV = d(\rho V) + P\,dV = d[(\rho+P)V] - V\,dP$$

$$= d[(\rho+P)V] - \frac{V}{T}(\rho+P)\,dT$$

$\mu \simeq 0$

$$\frac{\partial P}{\partial T} = \frac{\rho+P}{T}$$

using distributions functions

$$dS = \frac{1}{T}\,d[(\rho+P)V] - \frac{V}{T^2}(\rho+P)\,dT =$$

$$= d\left[\frac{\rho+P}{T}V\right]$$

Total entropy: $S = \dfrac{(\rho+P)}{T}V$

- $$\frac{dS}{dt} = \frac{d}{dt}\left(\frac{\rho+P}{T}V\right) = \frac{V}{T}\left(\frac{d\rho}{dt} + \frac{dP}{dt}\right) + (\rho+P)\left(\frac{dV}{dt}\frac{1}{T} - \frac{V}{T^2}\frac{dT}{dt}\right)$$

$$= \frac{V}{T}\underbrace{\left(\frac{d\rho}{dt} + \frac{1}{V}\frac{dV}{dt}(\rho+P)\right)}_{0} + \frac{V}{T}\underbrace{\left(\frac{dP}{dt} - \frac{\rho+P}{T}\frac{dT}{dt}\right)}_{0} = 0$$

$$\dot\rho + 3H(\rho+P) = 0 \qquad V \sim a^3 \qquad \frac{\partial P}{\partial T} = \frac{(\rho+P)}{T}$$

- Entropy density: $s = \dfrac{S}{V} = \dfrac{(\rho+P)}{T}$

$$s = \sum_i \frac{\rho_i + P_i}{T_i} = \frac{2\pi^2}{45} g_{*s}(T)\,T^3$$

$$\rho = \frac{\pi^2}{30} g T^4$$

$$P = \frac{1}{3}\rho$$

$g_{*s}(T)$: entropic degrees of freedom

ⓐ If all particles are in thermal equilibrium:

$$g_{*s}^{th}(T) = \sum_i g_b + \frac{7}{8}\sum_i g_f = g_*^{th}(T)$$

ⓑ If some particles are not in thermal equilibrium:

$$g_{*s}^{dec}(T) = \sum_{i=b} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=f} g_i\left(\frac{T_i}{T}\right)^3 \neq g_*^{dec}(T)$$

Energetic and entropic degrees of freedom are the same only when all the relativistic species are in thermal equilibrium (they have the same temperature)

⊙ $s \propto a^{-3}$

$n_i \propto a^{-3}$

$$\boxed{N_i = \frac{n_i}{s}}$$

→ is constant if particles are not created or destroyed

⊙ $$\frac{dS}{dt} = 0 \Rightarrow \boxed{g_{*s} T^3 a^3 = const} \rightarrow \boxed{T \propto \frac{1}{a\, g_{*s}^{-1/3}}}$$

when a particle becomes effectively massive, it "heats" the universe

(T decreases slightly 'slower' than $T \propto a^{-1}$)

Summary

$T_i \gg m_i$ bosons fermions

$$\rho_r = \frac{\pi^2}{30} g_*(T) T^4$$ energy density rel. sp.

$$g_*(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4$$

$$s_r = \frac{2\pi^2}{45} g_{*s}(T) T^3$$ entropy density

$$g_{*s}(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3$$

T: photon temperature

- if $T_i = T \rightarrow g_*(T) = g_{*s}(T)$
- if $T_i \neq T \rightarrow g_*(T) \neq g_{*s}(T)$

$$\frac{dS}{dt} = 0 \Longrightarrow T \propto \frac{1}{a\, g_{*s}^{1/3}(T)}$$

$t \longleftrightarrow T$

$$M_p = \frac{1}{\sqrt{G}}$$

$$H^2 = \frac{8\pi G}{3}\rho \simeq \frac{8\pi G}{3}\rho_r = \frac{8\pi G}{3}\left(\frac{\pi^2}{30}\right) g_*(T) T^4 \Longrightarrow$$

$$\sqrt{\frac{8\pi^3}{90}} \sim 1.66$$

$$\Rightarrow \underbrace{H}_{\frac{1}{a}\frac{da}{dt} = -\frac{1}{T}\frac{dT}{dt}} \simeq 1.66\, g_*^{1/2} \frac{T^2}{M_{pl}} \Rightarrow \boxed{\frac{T}{1\,\mathrm{MeV}} \simeq 1.5\, g_*^{-1/4} \left(\frac{1\,\mathrm{sec}}{t}\right)^{1/2}}$$

$t = 1\,\mathrm{sec} \longrightarrow T = 1\,\mathrm{MeV}$

(2.1.4) Neutrino decoupling $T \sim 1\,\mathrm{MeV} \longleftrightarrow t \sim 1\,s$

At $T \simeq 1-10\,\mathrm{MeV}$, only γ, $e^{\pm}$, ν are relativistic.

Neutrinos are coupled to the thermal plasma:

$$\left\{\begin{array}{l} \nu_e + \bar{\nu}_e \longleftrightarrow e^+ + e^- \\ e^- + \bar{\nu}_e \longleftrightarrow e^- + \bar{\nu}_e \end{array}\right\} \qquad \sigma \sim G_F^2 T^2 \qquad G_F \sim \frac{\alpha}{M_W^2}$$

$$\Gamma \sim n\sigma \underbrace{\cancel{v}}_{\sim c \sim 1}{}^{1} \underset{\underset{n \sim T^3}{\uparrow}}{=} G_F^2 T^5$$

$$\frac{\Gamma}{H} \underset{\underset{H \sim \frac{T^2}{M_{pl}}}{\uparrow}}{\sim} \left(\frac{T}{1\,\mathrm{MeV}}\right)^3 << 1$$

$T >> 1\,\mathrm{MeV} \Rightarrow$ neutrinos are in therm. equil.,
$T \lesssim 1\,\mathrm{MeV} \Rightarrow$ neutrinos decouple.

$T_{dec} \sim 1\,\mathrm{MeV} \rightarrow$ in reality $\boxed{T_{dec} \sim 0.8\,\mathrm{MeV}}$

→But neutrinos are still relativistic, they contribute to g_* & g_{*s}

$$p \sim \frac{1}{a} \qquad n_\nu \propto a^{-3} \int d^3q \frac{1}{e^{\frac{q}{aT_\nu}} + 1}$$

$$f(p) \sim \frac{1}{e^{p/T} \pm 1} \qquad q = ap$$

$$\boxed{T_\nu \sim \frac{1}{a}}$$

$$\boxed{T \sim \frac{1}{a\, g_{*s}(T)^{-1/3}}}$$

(2.1.5) Electron–positron annihilation $T \simeq m_e \simeq 0.51\,\mathrm{MeV}$

-electrons become non-relativistic thermal: $e^+, e^-, \gamma, \cancel{\nu}$

There is a change in $g_{*s}^{\boxed{th}}$:

$$g_{*s}^{th} = \left\{\begin{array}{ll} 2 + \frac{7}{8} \times 4 & T \gtrsim m_e \\ 2 & T < m_e \end{array}\right.$$

$$\boxed{T^{(T \gtrsim m_e)} = \left(\frac{4}{11}\right)^{1/3} T^{(T \lesssim m_e)}}$$

entropy conservation $\Rightarrow g_{*s}(T)\, T^3 a^3 = \text{const}$

$$\underbrace{g_{*s}^{(T \gtrsim m_e)}}_{\frac{11}{2}} T^{(T \gtrsim m_e)^3} = \underbrace{g_{*s}^{(T \lesssim m_e)}}_{2} T^{(T \lesssim m_e)^3}$$

For $T \lesssim m_e \Rightarrow \boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$

Today: $T_\gamma = T_{CMB} = 2.73K$

$$T_\nu = 2.73\left(\frac{4}{11}\right)^{1/3} = 1.95K$$

⊛ For $T << m_e$:

$$g_a(T) = 2 + \frac{7}{8} \times 2 \times N_{eff} \times \left(\frac{4}{11}\right)^{4/3} = 3.36$$

$$g_{as}(T) = 2 + \frac{7}{8} \times 2 \times N_{eff} \times \left(\frac{4}{11}\right)^{1} = 3.94$$

N_{eff}: effective number of neutrino species $\Rightarrow N_{eff} \simeq 3$

$N_{eff} = 3.046$ (SM) [$\neq 3$ because neutrino decoupling is not instantaneous, and it is not finish at $e^- + e^+$ annihilation. Extra entropy is transfered to the neutrinos]

Planck experiment (2018): $N_{eff} = 2.99 \pm 0.17$

⊛ <u>Photon background</u> (exercises)

$T_\gamma = 2.73K$ → $n_\gamma = \frac{2}{\pi^2} \zeta(3) T_\gamma^3 \simeq 410 \frac{\text{photons}}{\text{cm}^3}$

→ $\rho_\gamma = \frac{\pi^2}{30} g T_\gamma^4 = 4.6 \times 10^{-34} \, g \, cm^{-3}$

↳ $\Omega_\gamma h^2 = 2.5 \times 10^{-5} << 1$

• <u>Neutrino background</u>

$$n_\nu = n_\gamma \times \left(\frac{4}{11}\right) \times \frac{3}{4} \times N_{eff} = 112 \frac{\text{neutrinos}}{\text{cm}^3}$$

$\rho_\nu = \dots \Rightarrow \Omega_\nu h^2 \simeq 1.7 \times 10^{-5} << 1$ $(m_\nu = 0)$

⊛ Neutrino oscillations $\Rightarrow \sum m_{\nu,i} > 0.06 \, eV$ (lower bound)

↳ $\Omega_\nu h^2 \simeq \frac{\sum m_{\nu,i}}{94 \, eV}$

Planck (2018): $\sum m_{\nu,i} < 0.12 \, eV$ (upper bound)

2.2 EVOLUTION BEYOND EQUILIBRIUM

2.2.1. Boltzmann equations
2.2.2. Dark matter freeze-out
2.2.3. Recombination and photon decoupling
2.2.4. Nucleosynthesis

① BOLTZMANN EQUATIONS

- No interactions: Particle number is conserved. Physical volume $\sim a^{-3}$

  n_i = number density $\Rightarrow$ $\boxed{n_i \propto a^{-3}}$

  $$\boxed{\frac{dn_i}{dt} + 3\frac{\dot a}{a} n_i = 0}$$

- With interactions: $\Rightarrow$ Boltzmann equations

  $$\boxed{\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = \underbrace{C_i[\{n_j\}]}_{\text{collision terms}}}$$

If $1+2 \rightleftarrows 3+4 \Rightarrow \frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = \underbrace{-\alpha n_1 n_2}_{\text{DESTRUCTION TERM}} + \underbrace{\beta n_3 n_4}_{\text{CONSTRUCTION TERM}}$

α: thermally averaged cross section $\alpha = \underbrace{\langle \sigma v \rangle}_{\text{averaged over velocities}}$

$\beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \alpha$, such that $C_i = 0$ in equilibrium

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle \sigma v \rangle \left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} n_3 n_4 \right]$$

$N_i \equiv n_i / s$

$N_i \propto n_i a^3$; $\Gamma_1 = n \langle \sigma v \rangle$

$\left(\begin{array}{l} s \sim a^{-3} \\ n_i \sim a^{-3} \text{ (no interactions)} \\ N_i \rightarrow \text{const} \end{array} \right)$

$$\frac{d \ln N_1}{d \ln a} = \frac{-\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2} \right]$$

$$\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\underbrace{\left[1-\left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq}\frac{N_3 N_4}{N_1 N_2}\right]}_{\Delta}$$

$1+2 \rightleftarrows 3+4$

Γ_1/H: interaction efficiency

if $\frac{\Gamma_1}{H} \gg 1 \longrightarrow$

- if $N_1 \gg N_1^{eq}$, $N_2 \gg N_2^{eq}$, $N_3 \ll N_3^{eq}$, $N_4 \ll N_4^{eq}$ $\Rightarrow \Delta > 0$
- if $N_1 \ll N_1^{eq}$, $N_2 \ll N_2^{eq}$, $N_3 \gg N_3^{eq}$, $N_4 \gg N_4^{eq}$ $\rightarrow \Delta < 0$

$$\boxed{\begin{aligned} N_1 &\to N_1^{eq}\\ N_2 &\to N_2^{eq}\\ N_3 &\to N_3^{eq}\\ N_4 &\to N_4^{eq}\end{aligned}}$$

- if $\frac{\Gamma_1}{H} \ll 1$ Species 1 : $N_1 \longrightarrow \text{constant} \neq N_1^{eq}$

$\boxed{\text{Freeze-out}}$

Ⓔ Boltzmann equations : [2.2.1]

$1,2 \to \chi, \bar{\chi}$
$3,4 \to \ell, \bar{\ell}$
$N_\ell = N_\ell^{eq}$

$$\underset{N_1}{1} + \underset{N_2}{2} \rightleftarrows \underset{N_3}{3} + \underset{N_4}{4}$$

interaction rate

$$\hookrightarrow \frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\boxed{eq}} \frac{N_3 N_4}{N_1 N_2}\right]$$

equilibrium

$\Gamma_1 = n_2 \langle \sigma v \rangle$: interaction rate

- Γ_1/H : interaction efficiency

$\Gamma_1/H \gg 1$ ← [early times]

$\hookrightarrow \dfrac{d\ln N_1}{d\ln a} = 0 \longrightarrow N_i = N_i^{eq} \quad (i: 1,2,3,4)$

$\hookrightarrow N_1 \gg N_1^{eq} \Rightarrow \dfrac{d\ln N_1}{d\ln a} \ll 0 \to N_1 \to N_1^{eq}$

$\hookrightarrow N_1 \ll N_1^{eq} \Rightarrow \dfrac{d\ln N_1}{d\ln a} \gg 0 \to N_1 \to N_1^{eq}$

- $\underline{\underline{\Gamma_1/H \ll 1}}$ [later times]

$\dfrac{d\ln N_1}{d\ln a} \to 0$: $N_1 \longrightarrow N_1^{\infty}$ (const) $\neq N_1^{eq}$

[FREEZE OUT]

(2.2.2) Dark matter freezeout

- Let us show that Boltzmann eqs can provide a mechanism to explain Dark Matter.

HYPOTHESIS: DM is a <u>WIMP</u> (Weak Interacting Massive Particle)

DM: X

Assumptions:

- X interacts with charged light particles (e.g charged leptons)

$$X + \bar{X} \longleftrightarrow \ell + \bar{\ell}$$

- No initial asymmetry $N_\chi = N_{\bar{\chi}}$
- Leptons tightly coupled to thermal plasma: $N_\ell = N_\ell^{eq}$
- Electric neutral : $N_\ell = N_{\bar{\ell}}$

$$\frac{d\ln N_x}{d\ln a} = \frac{-\Gamma_x}{H}\left[1-\frac{(N_x^{eq})^2}{N_x^{2}}\right] \qquad s=\frac{2\pi^2}{45}g_{*s}T^3$$

- $\Gamma_x \doteq n_x\langle\sigma v\rangle \underset{\substack{\uparrow \\ n_x = N_x \cdot s}}{=} N_x\, s\,\langle\sigma v\rangle \overset{\checkmark}{\simeq} \frac{2\pi^2}{45}g_{*s}T^3\langle\sigma v\rangle$ $\quad \nwarrow g_{*s}(T)\simeq \text{const}$

- $x=\frac{M_x}{T} \longrightarrow \frac{d\ln N_x}{d\ln a} \underset{\substack{\uparrow \\ T\propto\frac{1}{a}}}{=} \frac{x}{N_x}\frac{dN_x}{dx}$

- RD: $H=\frac{H(T=M_x)}{x^2}$

$$\boxed{\frac{dN_x}{dx} = -\frac{\lambda}{x^2}\left[N_x^2-(N_x^{eq})^2\right]} \qquad \lambda=\frac{2\pi^2}{45}g_{*s}\frac{M_x^3\langle\sigma v\rangle}{H(M_x)}\simeq \text{const}$$

At very late times $N_x \gg N_x^{eq}$

$N_x^{f} \gg N_x^{\infty}$

$$\int_{N_x^{f}}^{N_x^{\infty}}\frac{dN_x}{dx} \simeq \int_{x_f\sim 8.5}^{x=\infty}\frac{-\lambda N_x^2}{x^2} \Rightarrow \frac{1}{N_x^{\infty}}-\cancel{\frac{1}{N_x^{f}}} = \frac{\lambda}{x_f} \rightarrow \boxed{N_x^{\infty}\ \frac{x_f}{\lambda}}$$

if $\lambda\uparrow$, $N_x^{\infty}\downarrow$

- What is the value of $\langle\sigma v\rangle$ that could explain dark matter?

$\Omega_x \overset{?}{\simeq} 0.2$

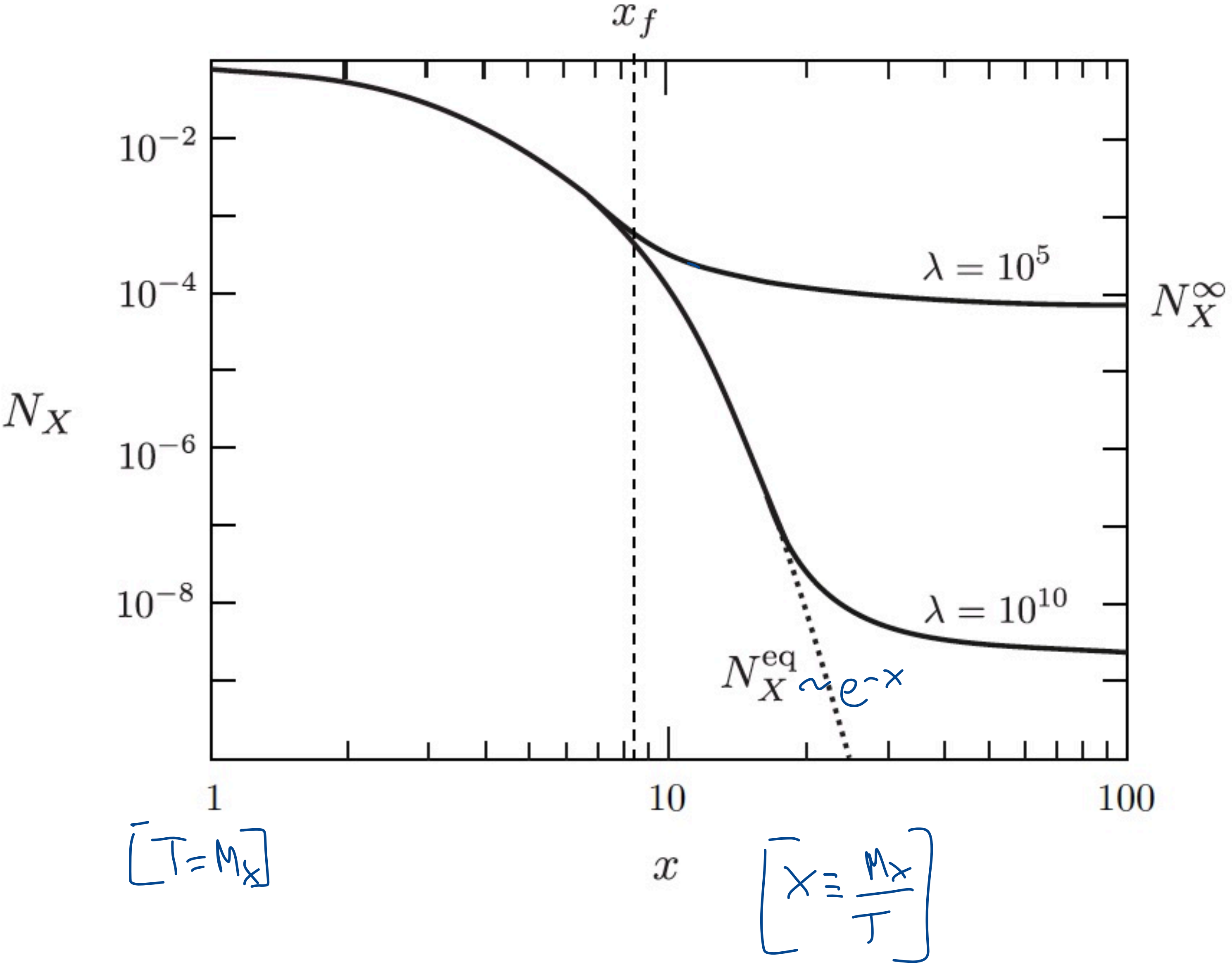
$$\Omega_x = \frac{\rho_{x,0}}{\rho_{crit,0}} = \frac{M_x n_{x,0}}{3M_{Pl}^2H_0^2} = \dots \rightarrow$$

$$\Rightarrow \overset{h\sim 0.7}{\Omega_x h^2} \sim 0.1\overset{\sim 1}{\left(\frac{x_f}{10}\right)}\overset{\sim 1}{\left(\frac{10}{\underset{\substack{\uparrow \\ g_*\sim 10}}{g_*(M_x)}}\right)^{1/2}}\frac{10^{-8}\,GeV^{-2}}{\langle\sigma v\rangle}$$

$\hookrightarrow \langle\sigma v\rangle \sim 10^{-4}\,GeV^{-1} \sim 0.1\sqrt{G_F}$

WIMP miracle.

$$\frac{dN_x}{dx} = -\frac{\lambda}{x^2}\left(N_x^2-(N_x^{eq})^2\right)$$



$[T = M_x]$

$\left[x \equiv \frac{M_x}{T}\right]$

2.2.3. Recombination and photon decoupling

- Recombination: Formation of the first atoms, around $T \sim 1\,\mathrm{eV}$

$t \sim 300000$ years

- At $T \gg 1\,\mathrm{eV}$: $e^- + p^+ \longleftrightarrow H + \gamma$ (Coulomb scattering)

For $i = \{p, e, H\}$, $T < m_i \Rightarrow n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}}$

$\mu_e + \mu_p = \mu_H \quad (\mu_\gamma = 0)$

$$\left(\frac{n_H}{n_e n_p}\right)_{eq} = \frac{g_H}{g_e g_p}\left(\frac{m_H}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} e^{B_H/T}$$

$B_H = m_p + m_e - m_H = 13.6\,\mathrm{eV}$ Binding energy of hydrogen

- $g_p = 2$, $g_e = 2$, $g_H = 4$; $n_e = n_p$ (not electrically charged) ; $m_H \simeq m_p$ (only in prefactor)

$$\left(\frac{n_H}{n_e^2}\right)_{eq} = \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}$$

Free electron ratio: $X_e = \frac{n_e}{n_b} \leftarrow$ baryon number density

$n_b = n_p + n_H \underset{\substack{\uparrow \\ \text{neutrality}}}{=} n_e + n_H$

$$X_e = \frac{n_e}{n_e + n_H}$$

- Saha equation:

$$\rightarrow \frac{1 - X_e}{X_e^2} = \frac{1 - \frac{n_e}{n_b}}{\left(\frac{n_e}{n_b}\right)^2} = \frac{n_b^2 - n_e n_b}{n_e^2} = \frac{n_b(n_b - n_e)}{n_e^2} = \frac{n_b n_H}{n_e^2}$$

$$\left[n_b = \frac{n_b}{n_\gamma} n_\gamma = \eta_b \times \frac{2\zeta(3)}{\pi^2} T^3 \right]$$

$\rightarrow \eta_b$: baryon-to-photon ratio $\rightarrow \eta_b = 5.5 \times 10^{-10} \left(\frac{\Omega_b h^2}{0.02}\right)$

$$\boxed{\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2} \eta_b \left(\frac{2\pi T}{m_p}\right)^{3/2} e^{B_H/T}}$$

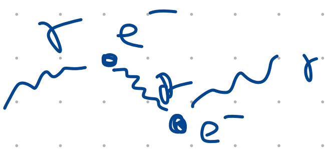
$\rightarrow \underline{\underline{X_e(T)}}$

Saha equation

- We define recombination when $X_e = 0.1$ $\Rightarrow$

$$\boxed{\begin{array}{l} T_{rec} \simeq 0.3 eV \simeq 3600K \\ \quad \updownarrow \\ z_{rec} \simeq 1320 \\ \quad \updownarrow \\ t_{rec} = 290000 \text{ years} \end{array}}$$

- Photon decoupling



- $e^- + \gamma \leftrightarrow e^- + \gamma$ Compton scattering
  - At $T \gg 1 eV$, photons are strongly coupled to the primordial plasma via interaction with electrons $\Rightarrow$ Universe is invisible
  - At $T \sim 1 eV$, electron density decreases; photons start propagating freely $\Rightarrow$ universe becomes visible.
  - These photons observed now as CMB $\rightarrow$ "last-scattering surface"

Γ_γ, H

- At $T \gg 1 eV$ $\quad \frac{\Gamma_\gamma}{H} \gg 1 \rightarrow$ photons are in thermal equil.
- $\Gamma_\gamma \simeq n_e \underset{\sim 2.10^{-3} MeV^{-2}}{\sigma_T}$; $\quad n_e \downarrow \rightarrow$ eventually $\Gamma_\gamma \ll H$

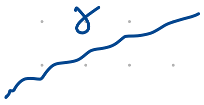
- Decoupling time: $\Gamma_\gamma(t_{dec}) = H_\gamma(t_{dec})$

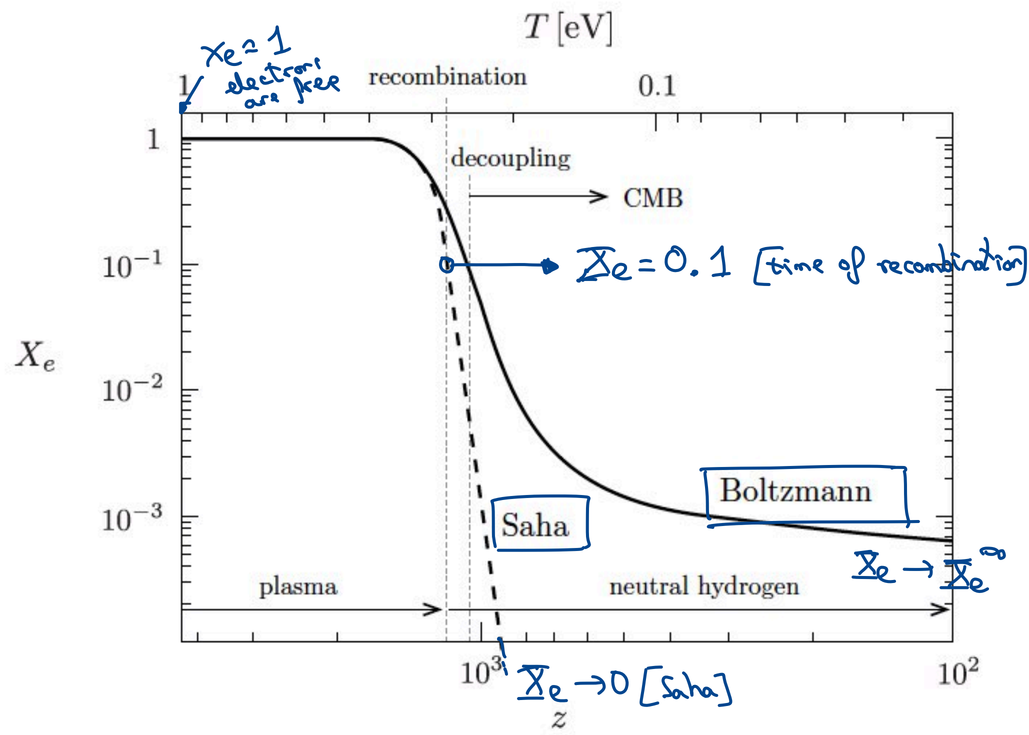
$$\hookrightarrow \Gamma_\gamma \simeq n_e \sigma_T \underset{\overset{\uparrow}{X_e = \frac{n_e}{n_b}}}{=} n_b X_e \sigma_T \underset{\overset{\nearrow}{n_b = \frac{2\zeta(3)}{\pi^2}\eta_b T_{dec}^3}}{=} \frac{2\zeta(3)}{\pi^2} \eta_b \sigma_T X_e(T_{dec}) T_{dec}^3 \quad \downarrow$$

$$\hookrightarrow H(t_{dec}) = H_0 \sqrt{\Omega_m} \left[\frac{T_{dec}}{T_0}\right]^3$$

$$X_e(T_{dec}) T_{dec}^{3/2} = \frac{\pi^2}{2\zeta(3)} \frac{H_0\sqrt{\Omega_m}}{\eta_b \sigma_T T_0^{3/2}} \nearrow$$

$$\boxed{\begin{array}{l} T_{dec} \sim 0.27 eV \\ \quad \updownarrow \\ z_{dec} \sim 1100 \\ \quad \updownarrow \\ t_{dec} \sim 380000 \text{ years} \end{array}}$$





$X_e \simeq 1$ electrons are free

$X_e = 0.1$ [time of recombination]

$X_e \to X_e^{\infty}$

$X_e \to 0$ [Saha]

- Big Bang nucleosynthesis

$T \sim 1$ MeV
$t \sim 330$ sec

⊛ Formation of light atomic nuclei (H, He, Li, Be)

⊛ Full computation for particle numbers are obtained by solving Boltzmann eqs.

⊛ Simplifications $\Rightarrow$ We only consider H (p^+, D), He (^{3}He, ^{4}He)

⊛ Prediction: $\boxed{\dfrac{n_{He}}{n_H} \sim \dfrac{1}{16}}$

- $\{n, p\} \Rightarrow$ At $T >> 1$ MeV:

$$n + \nu_e \longleftrightarrow p^+ + e^- \quad (\beta\text{-decay})$$
$$n + e^+ \longleftrightarrow p^+ + \bar{\nu}_e \quad (\text{inverse } \beta\text{-decay})$$

$$(T < m_i) \longrightarrow n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}} \xrightarrow[\substack{\mu_\nu \simeq \mu_e \simeq 0 \\ \mu_n = \mu_p}]{} \Rightarrow \left(\frac{n_n}{n_p}\right)_{eq} = \cancelto{\sim 1}{\left(\frac{m_n}{m_p}\right)^{3/2}} e^{-(m_n - m_p)/T} \Rightarrow$$

$$\Rightarrow \left(\frac{n_n}{n_p}\right)_{eq} \simeq e^{-Q/T}$$

$$Q \equiv m_n - m_p = 1.30 \text{ MeV}$$

$$\boxed{\begin{array}{l} X_n = \dfrac{n_n}{n_n + n_p} \\ \text{neutron fraction} \end{array}} \rightarrow \boxed{X_n^{eq}(T) = \frac{e^{-Q/T}}{1 + e^{-Q/T}}}$$

⊛ Neutrinos decouple at $T_f \sim T_{dec} \sim 0.8$ MeV

$$\hookrightarrow X_n^{eq}(0.8 \text{ MeV}) = 0.17 \sim \frac{1}{6} \equiv X_n^\infty$$

[using Boltzmann eq. → $X_n^\infty \sim 0.15$ for neutron freeze-out]

⊛ Neutrons are unstable $\tau_n = 881.5 \pm 1.5$s lifetime

$$X_n(t) = X_n^\infty \cdot e^{-t/\tau_n} = \frac{1}{6} e^{-t/\tau_n}$$

Nuclear reactions involving three or more incoming nuclei are suppressed because the temperature and density are too low. Deuterium must form first.

- **Deuterium formation**

$n + p^+ \longleftrightarrow D + \gamma$ $\qquad m_D \simeq 2m_p$

$$\left(\frac{n_D}{n_n n_p}\right)_{eq} = \frac{3}{4}\left(\frac{m_D}{m_n m_p}\frac{2\pi}{T}\right)^{3/2} e^{-(m_D - m_n - m_p)/T} \overset{\int}{\simeq}$$

$$\simeq \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T} = \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T}$$

$$\left[n_n \sim n_b = \eta_b \cdot n_\gamma = \eta_b \frac{2\zeta(3)}{\pi^2} T^3\right]$$

$B_D = m_n + m_p - m_D = 2.22 \text{ MeV}$
binding energy of deuterium

$$\left(\frac{n_D}{n_p}\right)_{eq} \simeq 1 \longrightarrow \boxed{T \sim 0{,}06 \text{ MeV}}$$

- **Helium formation**

$D + p^+ \longleftrightarrow {}^3He + \gamma$
$D + {}^3He \longleftrightarrow {}^4He + p^+$

$B_{He} > B_D \Rightarrow$ Helium forms immediately after deuterium:
DEUTERIUM BOTTLENECK

Binding energy of helium

$$\boxed{T_{nuc} \sim 0.06 \text{ MeV}} \longrightarrow t_{nuc} \sim 330 \text{ sec}$$

time of nucleosynthesis $\qquad \frac{T}{1 \text{MeV}} = 1.5\, g_*^{-1} \sqrt{\frac{1 \text{sec}}{t}}$

$$\frac{n_{He}}{n_H} \simeq \frac{n_{He}}{n_p} = \frac{\frac{1}{2} X_n(t_{nuc})}{1 - X_n(t_{nuc})} \sim \frac{1}{16}$$

$$\left(X_n(t_{nuc}) = \frac{1}{6} e^{-t_{nuc}/\tau_n} \sim \frac{1}{8}\right)$$

$$\boxed{\frac{m_{He}}{m_H} \simeq 4\frac{n_{He}}{n_H} \sim \frac{1}{4}}$$

- BBN depends on $\{ g_*, \tau_n, Q, \eta_b, G_N, G_F \}$

  BBN can be a probe of BSM physics.

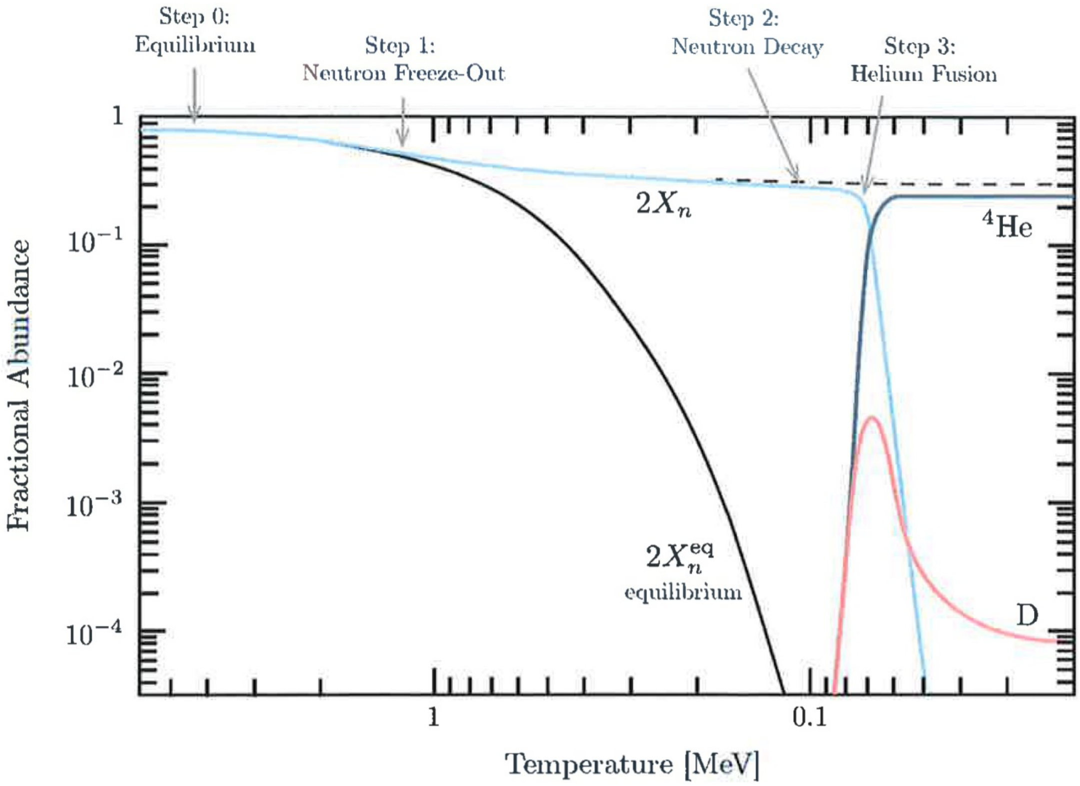
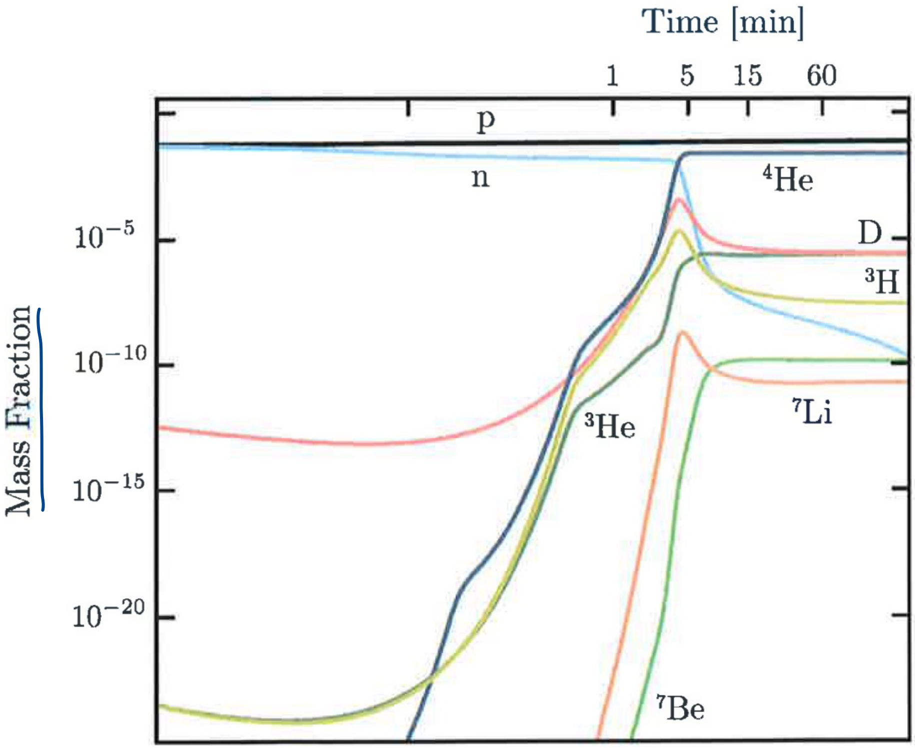
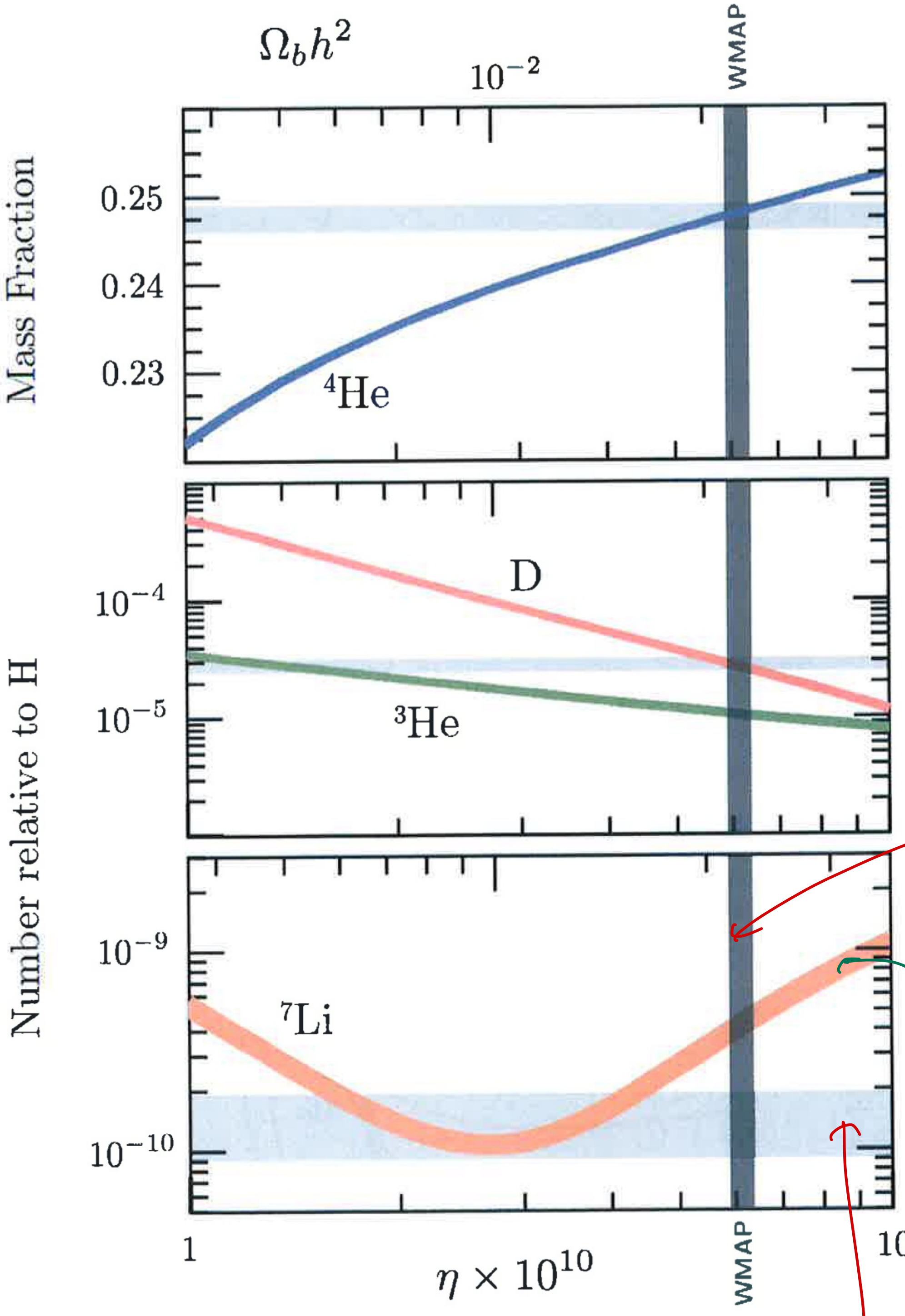


Figure 3.9: *Numerical results for helium production in the early universe.*



$$\rightarrow \boxed{\frac{m_{He}}{m_H} \sim \frac{1}{4}}$$

Figure 3.11: *Numerical results for the evolution of light element abundances.*



observations

predictions

observations

$\boxed{4}$ INFLATION

(4.1) Problems of the hot Big Bang theory

$\nearrow$ Horizon problem

$\searrow$ Flatness problem

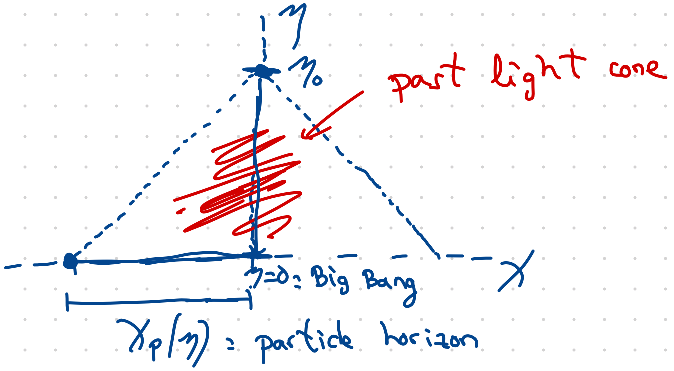
- Horizon problem

comoving radius

$K=0$

$\theta=\phi=0$

$$ds^2 = a^2(\eta)\left[d\eta^2 - d\chi^2\right] \longrightarrow \text{photons: } ds^2 = 0 \rightarrow \Delta\chi = \Delta\eta$$



- (Comoving) particle horizon

$$\chi_p(\eta_0) = \eta_0 - \eta_i \overset{dt = a\,d\eta}{=} \int_{t_i}^{t_0} \frac{dt}{a(t)} = \int_{a_i}^{a_0} \frac{da}{a\,\dot{a}} = \int_{\ln a_i}^{\ln a_0} (aH)^{-1}\, d\ln a = \dots$$

Comoving Hubble radius: $(aH)^{-1} = H_0^{-1} \cdot a^{\frac{1}{2}(1+3w)}$

$$\dots = \frac{2H_0^{-1}}{(1+3w)}\left[a_0^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] = \eta_0 - \eta_i$$

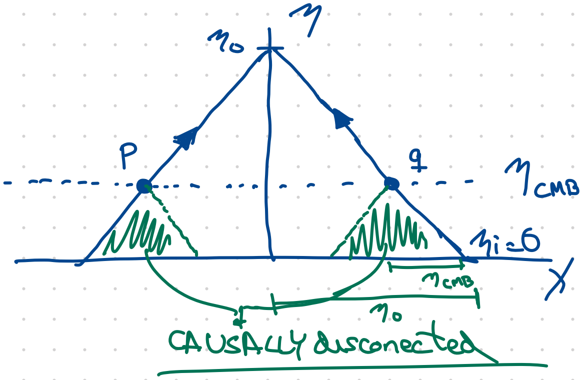
- If $w > -\frac{1}{3}$ (e.g. MD, RD) $\Longrightarrow \lim\limits_{a_i \to 0} \eta_i \longrightarrow 0$

If $w > -\frac{1}{3} \Rightarrow \chi_p(\eta) = \frac{2H_0^{-1}}{(1+3w)} a^{\frac{1}{2}(1+3w)} = \frac{2}{(1+3w)}(aH)^{-1}$

$\uparrow$

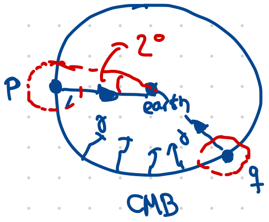
$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}$

in standard cosmology $w > -\frac{1}{3}$ $\Longrightarrow$ $\boxed{\text{particle horizon } \chi_p(t) \sim \text{comoving Hubble radius } (aH)^{-1}}$



$T_p \simeq T_q \simeq 2{,}73K$

HORIZON PROBLEM



When we look at the CMB, all photons coming from angular positions with more than $\Delta\theta \sim 1° - 2°$ are causally disconnected.

- Computation number of disconnected regions in the CMB:

$$N \simeq \frac{A_1}{A_{tot}} \sim \frac{\cancel{4\pi}\,\eta_{CMB}^2}{\cancel{4\pi}\,\eta_0^2} \sim \left(\frac{\eta_{CMB}}{\eta_0}\right)^2 \simeq \mathcal{O}(10^3) \text{ regions}$$

$\uparrow$

$\eta_0 = 14.4$ GpC

$\eta_{CMB} = 284$ MpC

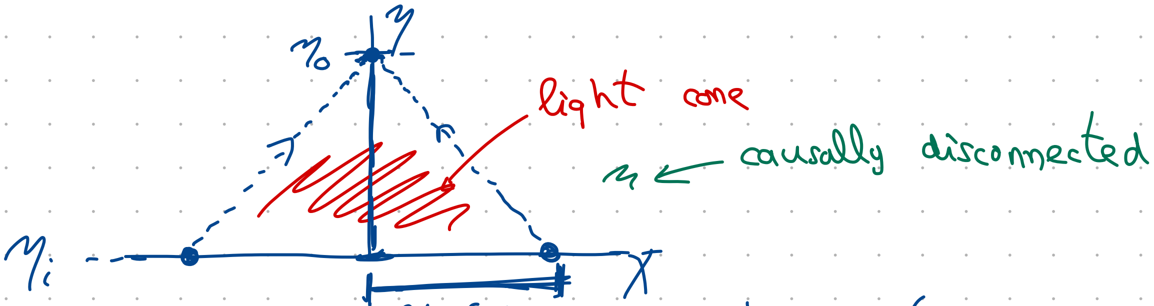
④ INFLATION

4.1. Problems of the hot Big Bang theory

→ horizon problem
→ flatness problem

① Horizon problem :

$$ds^2 = a^2(\eta)\,[d\eta^2 - d\chi^2] \qquad \text{photons: } d\eta = d\chi$$



$\chi_p(\eta)$ = particle horizon (comoving)

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t_0} \frac{dt}{a(t)} = \int_{\ln a_i}^{\ln a_0} (aH)^{-1}\, d\ln a \simeq$$

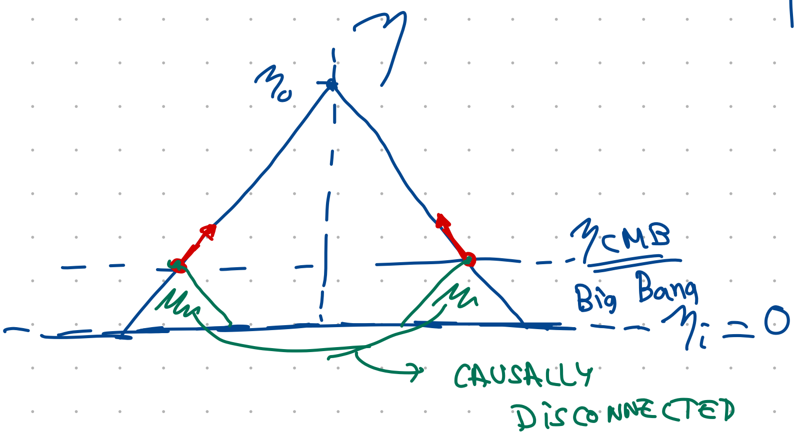
$$\boxed{(aH)^{-1} = H_0^{-1}\, a^{\frac{1}{2}(1+3w)}}$$

comoving Hubble radius

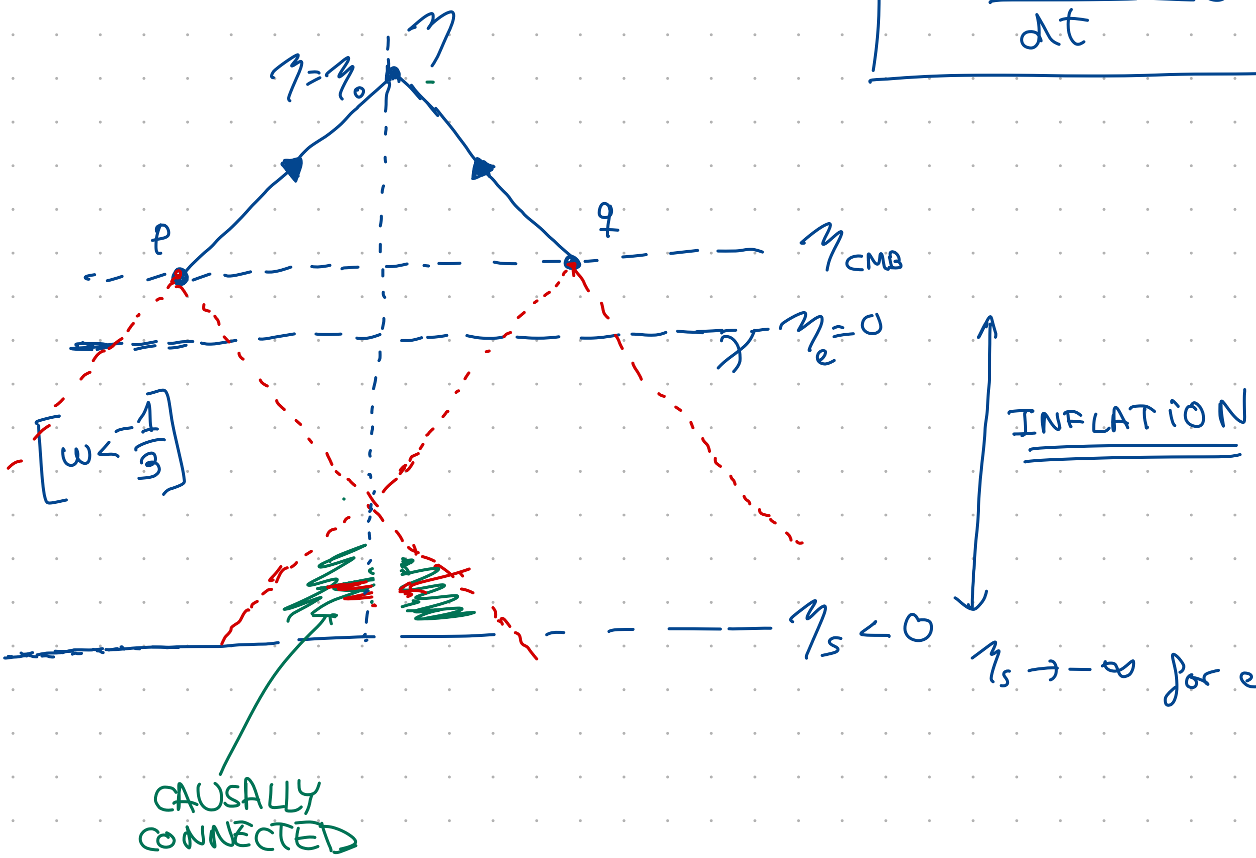
$$\simeq \frac{2H_0^{-1}}{(1+3w)}\left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] = \eta - \eta_i$$

- $\boxed{w > -\frac{1}{3}} \rightarrow \frac{1}{2}(1+3w) > 0$: $\boxed{\lim_{a_i \to 0} \eta_i \to 0 \qquad \frac{d(aH)^{-1}}{dt} > 0}$

e.g. RD ($w = \frac{1}{3}$) MD ($w = 0$)



- $\boxed{w < -\frac{1}{3}} \rightarrow \frac{1}{2}(1+3w) < 0$: $\boxed{\begin{array}{c} \lim_{a_i \to 0} \eta_i \rightarrow -\infty \\ \frac{d(aH)^{-1}}{dt} < 0 \end{array}}$



$\eta_s \rightarrow -\infty$ for eternal inflation

$w \simeq -1$ → $\boxed{\begin{array}{c} a(\eta) = -\frac{1}{H\eta} \\ H = \text{const} \end{array}}$; $\eta \in [-\infty, 0]$

Typical realization

(4.2) Definitions of inflation:

- Expansion sourced by a fluid with negative pressure, $w = \frac{P}{\rho} < -1/3$
- Shrinking comoving Hubble radius: $\frac{d}{dt}(aH)^{-1} < 0$ "Hubble sphere"
- Accelerated expansion: $\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{\dot{a}^2} < 0 \longrightarrow \boxed{\ddot{a} > 0}$
- Slowly-varying Hubble parameter: $\varepsilon \equiv \frac{-\dot{H}}{H^2}$

  $\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon) < 0$ $\boxed{\varepsilon \equiv \frac{-\dot{H}}{H^2} < 1}$
- "Constant" energy density: $\rho \propto a^{-3(1+w)} \simeq \text{const}$ $\underline{w \simeq -1}$

② Quasi de-Sitter expansion

If $\varepsilon = \frac{-\dot{H}}{H^2} \to 0$, $H \to \text{const}$ $\quad a(t) = e^{Ht}$

$$\boxed{ds^2 = dt^2 - e^{2Ht} d\vec{x}^2}$$ de sitter spacetime

It cannot be perfect "de Sitter" because inflation must eventually end.

4.1. [continuation]

- We require $|\eta_s| > |\eta_0|$ to solve horizon problem. $\frac{H_0}{H_e} = \left(\frac{a_0}{a_e}\right)^{-2}$ [assuming RD]

$$\frac{\eta_0}{\eta_s} = \frac{(a_0 H_0)^{-1}}{(a_s H_s)^{-1}} = \frac{(a_0 H_0)^{-1}}{(a_e H_e)^{-1}} \frac{(a_e \cancel{H_e})^{-1}}{(a_s \cancel{H_s})^{-1}} \underset{\uparrow\, H_s = H_e}{=} \frac{a_0}{a_e}\frac{a_s}{a_e} = \frac{T_e}{T_0}\frac{a_s}{a_e} =$$

$a \propto T^{-1}$

$$= 10^{29}\left(\frac{a_s}{a_e}\right).$$

$T_0 = 2.73\,K$

$T_e = 10^{16}\,GeV$

N [number of e-folds]

$a \sim e^N$

We require $\frac{a_e}{a_s} > 10^{29} \Rightarrow N = \log\left(\frac{a_e}{a_s}\right) \simeq 67$ e-folds.

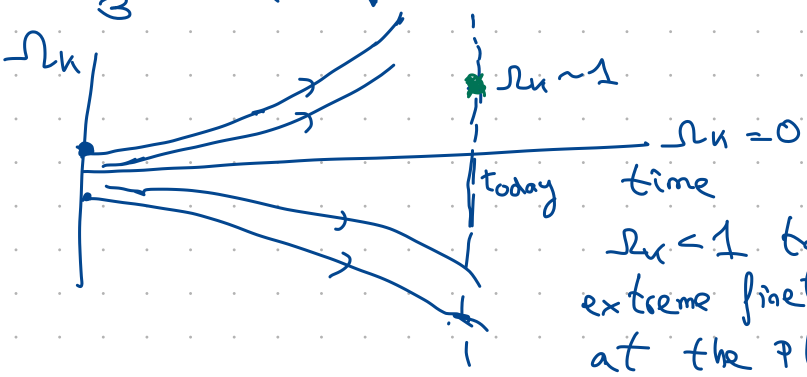
- Flatness problem:

We observe $\Omega_k \simeq 0$ $\quad \Omega = \Omega_m + \Omega_r = 1 - \Omega_k$ $\quad \left[\Omega_m, \Omega_r = \frac{8\pi G}{3H^2}\rho\right]$

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \Rightarrow |\Omega - 1|\rho a^2 = \frac{3|k|}{8\pi G} = \text{const}$$

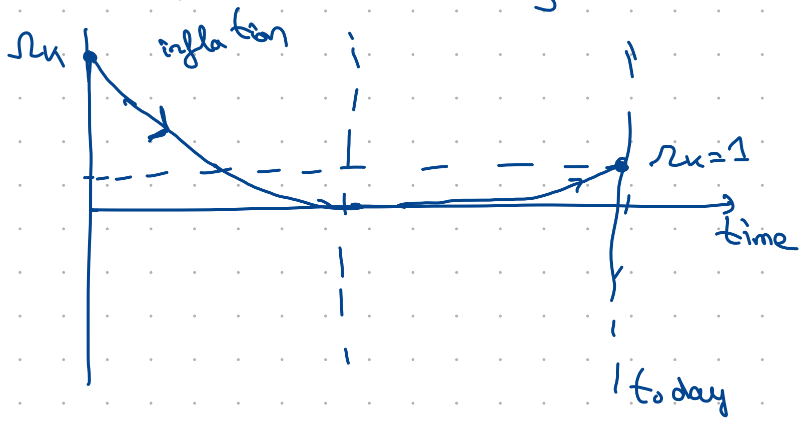
$$\rho \propto a^{-3(1+w)} \Rightarrow \rho a^2 \propto a^{-(3w+1)}$$

For $w > -\frac{1}{3} \Rightarrow \rho a^2 \downarrow \to |\Omega - 1| \uparrow \to \Omega_k = |1 - \Omega| \uparrow\uparrow$



$\Omega_k < 1$ today requires extreme finetuning $\Omega_k(t_{pl}) \lesssim 10^{-60}$ at the planck time.

During inflation: $w < -\frac{1}{3} \to \rho a^2 \uparrow \to |\Omega - 1| \downarrow \longrightarrow |\Omega_k| \downarrow\downarrow$



also solved with ~60-70 e-folds

4.3. Slow-roll inflation

A "toy" particle physics model of inflation:

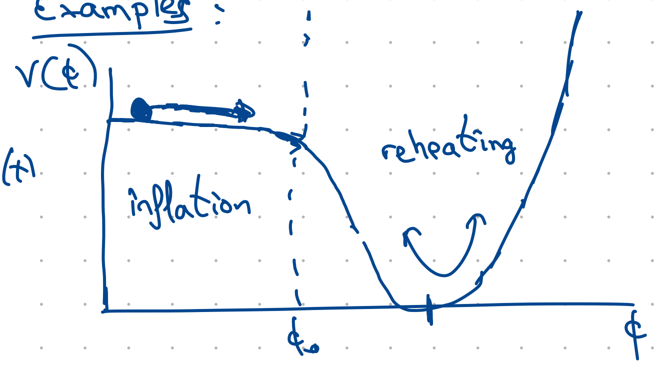
scalar field $\phi(\vec{x}, t)$: the inflaton,

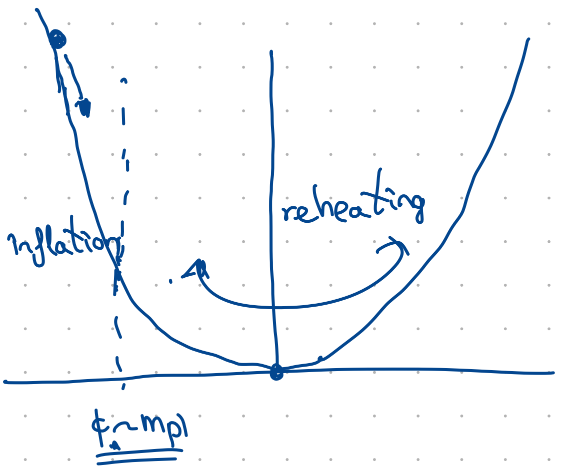
with potential energy $V(\phi)$.

Which properties must $V(\phi)$ have to sustain inflation?.

Examples:

$\phi \equiv \phi(t)$





e.g. $m^2\phi^2$, $\lambda\phi^4$

action $\quad S = \int \sqrt{-g}\, d^4x\, \mathcal{L} = \int \sqrt{-g}\, d^4x \left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$

$g = \det(g)$

stress-energy tensor $\quad T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,\mathcal{L})}{\delta g_{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right)$

⊛ For compatibility with FLRW ⇒ H & I ⇒ $\phi = \phi(\cancel{\vec{x}}, t)$

$\rho_\phi = T^0{}_0 = \underbrace{\frac{1}{2}\dot{\phi}^2}_{\text{kinetic energy}} + \underbrace{V(\phi)}_{\text{potential energy}}$ $\qquad P_\phi = -\frac{1}{3}\sum_i T^i{}_i = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

- 1st Friedmann eqn. $\Rightarrow H^2 = \frac{\rho_\phi}{3m_{pl}^2} \longrightarrow \boxed{H^2 = \frac{1}{3m_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)}$ (€)

$m_{pl} = \sqrt{\frac{\hbar c}{8\pi G}}$

$\frac{d(€)}{dt} \rightarrow 2H\dot{H} = \frac{\dot{\phi}}{3m_p^2}\left(\ddot{\phi} + \frac{\partial V}{\partial \phi}\right)$

combine

- 2nd Friedmann eqn $\Rightarrow \dot{H} = -\frac{(\rho_\phi + P_\phi)}{2m_{pl}^2} = -\frac{1}{2}\frac{\dot{\phi}^2}{m_{pl}^2}$

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0}$$ (R'€)

friction term

Klein-Gordon equation

$\left[\frac{\delta S}{\delta \phi} = 0\right.$
exercise $\left.\right]$

(4.3) Slow-roll inflation

⊕ Scalar field: inflaton $\phi(\vec{x},t)$ + $V(\phi)$

$$S=\int\sqrt{-g}\,d^4x\left\{\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-V(\phi)\right\}=\int\sqrt{-g}\,d^4x\,\mathcal{L}$$

$\phi=\phi(t)$ H + I

$\rho_\phi=T^0{}_0=\frac{1}{2}\dot\phi^2+V(\phi)$ $P_\phi=-T^i{}_i=\frac{1}{2}\dot\phi^2-V(\phi)$

1st $H^2=\frac{1}{3m_p^2}\rho_\phi \Rightarrow \boxed{H^2=\frac{1}{3m_p^2}\left(\frac{1}{2}\dot\phi^2+V(\phi)\right)}$ $\rightarrow$ ⊛ $\boxed{\ddot\phi+3H\dot\phi+\frac{\partial V}{\partial\phi}=0}$

Klein-Gordon eq.

2nd $\dot H=-\frac{(\rho_\phi+P_\phi)}{2m_p^2} \Rightarrow \dot H=-\frac{1}{2}\frac{\dot\phi^2}{m_p^2}$

→ Conditions to achieve inflation

① $w_\phi<-\frac{1}{3}$ $w_\phi=\frac{P_\phi}{\rho_\phi}=\frac{\cancel{\frac{1}{2}\dot\phi^2}-V(\phi)}{\cancel{\frac{1}{2}\dot\phi^2}+V(\phi)}\simeq-1 \Rightarrow \boxed{\dot\phi^2\ll|V(\phi)|}$

② We need condition "1" to be obeyed for at least 60 e-folds of expansion. $\boxed{\ddot\phi\ll 3H|\dot\phi|,\ V'(\phi)}$

Slow-roll parameters

$\boxed{\epsilon=\frac{\frac{1}{2}\dot\phi^2}{m_p^2H^2}\ll 1}$ $\boxed{\delta=\frac{-\ddot\phi}{H\dot\phi}\ll 1}$ $\eta=\frac{\dot\epsilon}{H\epsilon}=\dots$

$=2(\epsilon-\delta)\ll 1$

In this slow-roll regime; ⇒ $H^2\simeq\frac{V(\phi)}{3m_{pl}^2}$ ① ; $3H\dot\phi\simeq-V'(\phi)$ ②

$$\epsilon=\frac{\frac{1}{2}\dot\phi^2}{m_p^2H^2}\simeq\frac{m_p^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2=\frac{m_p^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2\equiv\epsilon_V$$

$\epsilon\simeq\epsilon_V$

$\eta_V\simeq\delta+\epsilon$

$V'(\phi)=V_{,\phi}$

$V''(\phi)=V_{,\phi\phi}$

$$\frac{d②}{dt}\rightarrow 3\dot H\dot\phi+3H\ddot\phi\simeq-V_{,\phi\phi}\dot\phi\rightarrow\delta+\epsilon\simeq M_{pl}^2\frac{V_{,\phi\phi}}{V}\equiv\eta_V$$

Potential must obey: $\epsilon_V\equiv\frac{M_p^2}{2}\left(\frac{\partial V/\partial\phi}{V(\phi)}\right)^2\ll 1$

$\eta_V\equiv M_p^2\left(\frac{\partial^2V/\partial\phi^2}{V(\phi)}\right)\ll 1$

- "Amount" of inflation:

$$N_{tot} = \int_{\textcircled{a_s}}^{\textcircled{a_e}} d\ln a = \int_{t_s}^{t_e} H(t)\, dt$$

← end of inflation

start of inflation

Number of e-folds of inflation

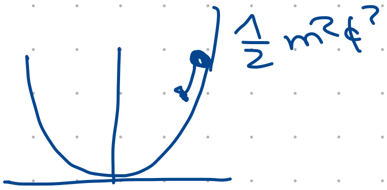
where $\varepsilon(t_s) = 1$; $\varepsilon(t_e) = 1$; $t_s < t < t_e \longrightarrow \varepsilon(t) < 1$ inflation

$$H\,dt = \frac{H}{\dot\phi}|d\phi| = \frac{1}{\sqrt{2\varepsilon}} \frac{|d\phi|}{M_{pl}} \simeq \frac{1}{\sqrt{2\epsilon_V}} \frac{|d\phi|}{M_{pl}}$$

$N_{tot} > 0$ $\left(\varepsilon = \frac{\frac{1}{2}\dot\phi^2}{m_p^2 H^2}\right)$

$$\boxed{N_{tot} = \int_{\phi_s}^{\phi_e} \frac{1}{\sqrt{2\epsilon_V(\phi)}} \frac{|d\phi|}{M_{pl}}}$$

We require $N_{tot} \gtrsim 60$



Example : $\boxed{V(\phi) = \frac{1}{2} m^2 \phi^2}$

$V'(\phi) = m^2\phi$
$V''(\phi) = m^2$ $\rightarrow \epsilon_V(\phi) = \frac{m_p^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 2\left(\frac{m_p}{\phi}\right)^2 = \eta_V(\phi)$

Inflation happens $|\epsilon_V|, |\eta_V| < 1 \rightarrow \boxed{\phi > \sqrt{2}\, M_{pl} \equiv \phi_e}$

"super-Planckian" amplitudes

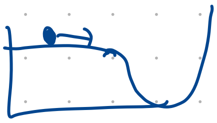
$$N(\phi) = \int_{\phi_e}^{\phi} \frac{d\phi}{m_p} \frac{1}{\sqrt{2\epsilon_V}} = \int_{\phi_e}^{\phi} \frac{d\phi}{m_{pl}} \frac{1}{2}\left(\frac{\phi}{m_p}\right) = \frac{\phi^2}{4 m_p^2} - \frac{1}{2}$$

$N(\phi) > 60 \rightarrow \phi = m_{pl}\sqrt{2N(\phi)} \simeq m_{pl}\sqrt{120} \sim 11\, m_{pl}$

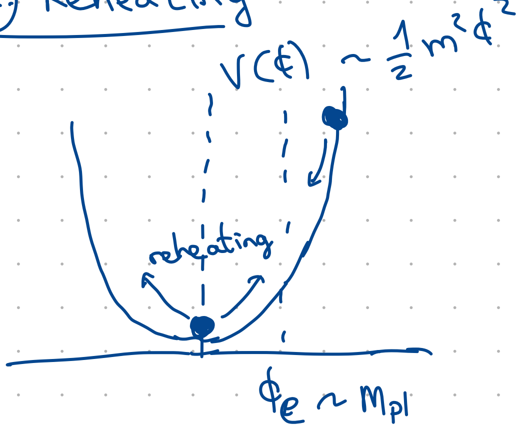
to do exercises

$\rightarrow V(\phi) = \frac{1}{n}\mu\phi^n \quad [\mu] = 4-n \qquad n = 2, 4, 6, 8$

$+ \; V(\phi) = V_0\left(1 - \frac{\phi^4}{\sigma^4}\right)^2 \rightarrow$



4.4. Reheating



→ After inflation, the energy of the inflaton must be transferred to other particles. This is called reheating.

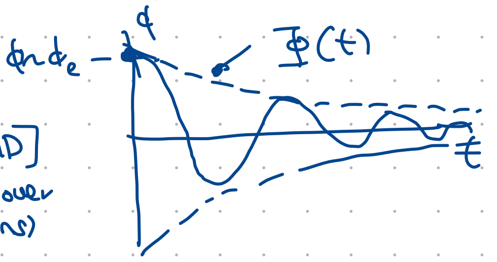
→ Details of reheating are very model-dependent, but one can typically identify similar processes.

$$\ddot{\phi} + \overset{\text{friction term}}{3H\dot{\phi}} + \overset{\text{"force term"}}{m^2\phi} = 0$$

→ When $\underline{\epsilon, \eta \ll 1}$ → slow-roll inflation ⟺ $\underline{H \gg m}$

→ when $\phi \simeq \phi_e \sim M_{pl}$ (i.e. $\epsilon, \eta \sim 1$) ⇒ $H \sim m$ → inflaton oscillates around the minimum of the potential

[exercise] → $\phi(t) = \underbrace{\Phi(t)}_{\text{decaying amplitude}} \sin(\overset{\text{frequency}}{m} t)$; $\Phi(t) \simeq \sqrt{\frac{8}{3}}\frac{M_{pl}}{mt}$



→ $a(t) \sim t^{2/3}$ [MD] (averaged over oscillations)

↳ $w \simeq 0$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \sim a^{-3}$$

⊛ Perturbative reheating

$\mathcal{L} \in g\phi\chi^2$; $h\bar{\psi}\psi\phi$

χ: spin-0 boson

ψ: spin-$\frac{1}{2}$ fermion

$\phi \longrightarrow \chi\chi$

$\phi \longrightarrow \bar{\psi}\psi$

$$\Gamma(\phi \to \chi\chi) = \frac{g^2}{8\pi m} \; ; \; \Gamma(\phi \to \bar{\psi}\psi) = \frac{h^2 m}{8\pi}$$

$$\Gamma = \Gamma(\phi \to \chi\chi) + \Gamma(\phi \to \bar{\psi}\psi)$$

$$\boxed{\ddot{\phi} + (3H + \underset{\text{friction term}}{\Gamma})\dot{\phi} + m^2\phi = 0} \longrightarrow \frac{d}{dt}(a^3\rho_\phi) = -\Gamma a^3 \rho_\phi$$

→ During the initial inflaton oscillations ⇒ $H \gg \Gamma$

→ Eventually $H \simeq \Gamma$ ⇒ reheating happens

$\rightarrow \; H(t) \sim \frac{2}{3t} \quad (\mathrm{MD})$

$\rightarrow \; H = \Gamma \longrightarrow t_r \sim \frac{2}{3}\Gamma^{-1} \xrightarrow[t=\sqrt{\frac{4}{3\rho}}M_{pl}]{} \rho(t_r) = 3\Gamma^2 M_{pl}^2$

- Assume instant thermalization: $\rho(t_r) = \frac{\pi^2}{30} g_*(T_r) T_r^4 \simeq 3\Gamma^2 M_{pl}^2$

$$\Longrightarrow \quad T_r \simeq \frac{3}{g_*^{1/4}(T_r)}\left(\frac{\Gamma}{\pi}M_{pl}\right)^{1/2} \underset{\tau}{\simeq} 0.5\sqrt{\Gamma M_{pl}}$$

$g_*(T_r) \sim 10^2 - 10^3$

REHEATING TEMPERATURE

- Preheating $\quad$ ('p')reheating

$\rightarrow$ Initial rapid out-of-equilibrium production of particles due to non-perturbative effects.

$$\boxed{\mathcal{L} \in \frac{1}{2}g^2\phi^2\chi^2} \Rightarrow \ddot{\chi} - \frac{1}{a^2}\nabla^2\chi + \cancel{3H}\dot{\chi} + \underbrace{g^2\phi^2}_{\text{effective mass, time-dependent!}}\chi = 0 \quad \Rightarrow$$

we neglect expansion

$$\Rightarrow \ddot{\chi}_k + \underbrace{\left(k^2 + g^2\Phi^2\sin^2(mt)\right)}_{\omega_k^2(t)}\chi_k = 0 \longrightarrow \boxed{\chi_k \sim e^{\mu_k t}}\;, \mathrm{Re}[\mu_k] > 0$$

$\uparrow$

$\frac{\dot{\omega}_k(t)}{\omega_k^2(t)} >> 1$

$\uparrow$

EXPONENTIAL
GROWTH

3 COSMOLOGICAL PERTURBATION THEORY

3.1 Metric and matter perturbations

- Universe is homogeneous and isotropic at large scales.
- In order to study structure formation, we need to introduce perturbations.

$$\begin{tikzcd} & \boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \arrow[dl] \arrow[dr] & \\ g_{\mu\nu}(\eta,\vec{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta,\vec{x}) & & T_{\mu\nu}(\eta,\vec{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta,\vec{x}) \end{tikzcd}$$

$$\bar{g}_{\mu\nu}(\eta) = a^2(\eta)\left(d\eta^2 - d\vec{x}^2\right) \qquad \bar{T}^0{}_0 = \bar{\rho}(\eta) \;;\; \bar{T}^i{}_i = -\bar{P}(\eta)$$

A) Metric perturbations:

$$ds^2 = a^2(\eta)\left[\underset{\substack{\uparrow\\ \text{scalar}}}{(1+2A)}d\eta^2 - \underset{\substack{\uparrow\\ \text{vector}}}{2B_i} dx^i d\eta - (\delta_{ij} + \underset{\substack{\uparrow\\ \text{tensor}}}{h_{ij}}) dx^i dx^j\right]$$

$$A = A(\eta,\vec{x}) \qquad B_i = B_i(\eta,\vec{x}) \qquad h_{ij} = h_{ij}(\eta,\vec{x})$$

⊛ Scalar-vector-tensor (SVT) decomposition:

- $B_i = \underset{\substack{\uparrow\\ \text{scalar}}}{\partial_i B} + \underset{\substack{\uparrow\\ \text{(transverse)}\\ \text{vector}}}{\hat{B}_i} \quad \Big| \quad \partial^i \hat{B}_i = 0$
- $h_{ij} = \underset{\substack{\uparrow\\ \text{scalar}}}{2C\delta_{ij}} + 2\partial_{<i}\partial_{j>}\underset{\substack{\uparrow\\ \text{scalar}}}{E} + \underset{\substack{\text{(transverse)}\\ \text{vector}}}{2\partial_{(i}\hat{E}_{j)}} + \underset{\substack{\text{(transverse}\\ \text{AND}\\ \text{traceless)}\\ \text{tensor}}}{2\hat{E}_{ij}} \quad \Big| \quad \partial^i \hat{E}_{ij} = 0 \;\; (\partial^j \hat{E}_{ij} = 0) \;\; \hat{E}^i{}_i = 0$

$$\hookrightarrow \partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2}\left(\partial_i \hat{E}_j + \partial_j \hat{E}_i\right)$$

$$\partial_{<i}\partial_{j>} \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E$$

Number of degrees of freedom:

- Scalars : 4×1 (A, B, C, E)
- Transverse vectors : 2×2 $(\hat{B}_i, \hat{E}_j)$
- TT tensors : 1×2 $(\hat{E}_{ij})$

10 d.o.f.

- In the SVT decomposition, Einstein's equations for scalar, vectors and tensors <u>do not mix at linear order</u> and can be treated separately.

⑤ Gauge fixing

Metric perturbations are not uniquely defined, they can change under coordinate transformations. There are only six real degrees of freedom,

$$X^\mu \longrightarrow \tilde{X}^\mu \equiv X^\mu + \xi^\mu(\eta,\vec{x}) \qquad \begin{aligned} &\xi^0 \equiv T \\ &\xi^i \equiv L^i = \partial^i L + \hat{L}^i \end{aligned}$$

Using invariance of spacetime interval:

$$ds^2 = g_{\mu\nu}(X)\, dX^\mu dX^\nu = \tilde{g}_{\alpha\beta}(\tilde{X})\, d\tilde{X}^\alpha d\tilde{X}^\beta$$

$$\downarrow$$

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}(x) \quad \begin{cases} \mu=\nu=0 \\ \mu=0;\nu=i \\ \mu=\nu=i \end{cases}$$

$$\boxed{\begin{aligned} &\bullet\ \text{Scalars: } A \rightarrow A - T'\mathcal{H}T \ ;\ B \rightarrow B + T - L' \ ; \\ &\qquad\qquad E \rightarrow E - L \ ;\ C \rightarrow C - \mathcal{H}T - \frac{1}{3}\nabla^2 L \\ &\bullet\ \text{Vectors: } \hat{B}_i \rightarrow \hat{B}_i - \hat{L}_i' \ ;\ \hat{E}_i \rightarrow \hat{E}_i - \hat{L}_i \\ &\bullet\ \text{Tensor: } \hat{E}_{ij} \longrightarrow \hat{E}_{ij} : \text{gauge-independent!} \end{aligned}}$$

- Solutions:

① Use gauge-invariant variables:

- Bardeen variables $\left\{ \begin{array}{l} \Psi \equiv A + \mathcal{H}(B - E') + (B - E')' \\ \Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E \\ \hat{\Phi}_i = \hat{E}_i' - \hat{B}_i \\ \hat{E}_{ij} \end{array} \right.$

$\updownarrow$ Bardeen potentials

$$\mathcal{H} = \frac{1}{a}\frac{da}{d\eta}$$

② Fix the gauge (i.e. fix T, L^i)

- Newtonian gauge (scalar part)

$$\left.\begin{array}{c} B = E = 0 \\ A = \Psi \\ C = -\Phi \end{array}\right\} \Rightarrow \boxed{ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right]}$$

- Spatially-flat gauge: $C = E = 0$ (used for inflation)
  $\hookrightarrow h_{ij} = 0$

B) Matter perturbations

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix} \Rightarrow \begin{aligned} T^{0}{}_{0} &= \bar{\rho}(\eta) + \delta\rho \\ T^{i}{}_{0} &= [\bar{\rho}(\eta) + \bar{P}(\eta)]\, \underset{\text{bulk velocity}}{v^{i}} \equiv \underset{\text{momentum density}}{q^{i}} \\ T^{i}{}_{j} &= -[\bar{P}(\eta) + \delta P]\,\delta^{i}{}_{j} - \underset{\text{anisotropic stress}}{\Pi^{i}{}_{j}} \end{aligned}$$

For perfect fluids $\Rightarrow \Pi^{i}{}_{j} = 0$

- Total stress-energy tensor: $\boxed{T_{\mu\nu} = \sum_a T_{\mu\nu}^{(a)} \; ; \; a = \gamma, \nu, c, b \ldots}$

$$\delta\rho = \sum_a \delta\rho_a \; ; \; \delta P = \sum_a \delta P_a \; ; \; q^{i} = \sum_a q^{i}_{(a)} \; ; \; \Pi^{ij} = \sum_a \Pi^{ij}_{(a)}$$

- Density contrast: $\boxed{\delta_a \equiv \frac{\delta\rho_a}{\rho_a} \; ; \; \delta = \frac{\delta\rho}{\rho}}$

$$\delta_a = \frac{\delta\rho_a}{\bar{\rho}_a + \delta\rho_a} \ll 1$$

⊛ SVT decomposition:

$$\begin{cases} \delta\rho \; ; \; \delta P \; ; \; q_i = \partial_i \underset{\text{scalar}}{q} + \underset{\text{vector}}{\hat{q}_i} \\ \Pi_{ij} = \partial_{<i}\partial_{j>} \underset{\text{scalar}}{\Pi} + \partial_{(i} \underset{\text{vector}}{\hat{\Pi}_{j)}} + \underset{\text{tensor}}{\hat{\hat{\Pi}}_{ij}} \end{cases}$$

10 dof.

⊛ Gauge fixing

$$X^{\mu} \longrightarrow \tilde{X}^{\mu} = X^{\mu} + \xi^{\mu}(\eta, \vec{x}) \qquad \xi^{0} = T, \; \xi^{i} = L^{i} = \partial^{i} L + \hat{L}^{i}$$

Using:
$$T^{\mu}{}_{\nu}(x) = \frac{\partial X^{\mu}}{\partial \tilde{X}^{\alpha}} \frac{\partial \tilde{X}^{\beta}}{\partial X^{\nu}} \tilde{T}^{\alpha}{}_{\beta}(\tilde{x})$$

$\Rightarrow$

$$\boxed{\begin{aligned} \delta\rho &\longrightarrow \delta\rho - T\bar{\rho}' \\ \delta P &\longrightarrow \delta P - T\bar{P}' \\ q_i &\longrightarrow q_i + (\bar{\rho} + \bar{P})\, L_i' \\ v_i &\longrightarrow v_i + L_i' \\ \Pi_{ij} &\longrightarrow \Pi_{ij} \quad \text{gauge-independent!} \end{aligned}}$$

- Gauge independent quantity: Δ

$$\bar{\rho}\Delta \equiv \delta\rho + \bar{\rho}'(v + B) \; ; \; v_i = \partial_i v$$

- Gauge fixing $<$
  - Uniform density gauge: $\delta\rho = 0$
  - Comoving gauge: $q = 0$

3.2. Equations of motion in the Newtonian gauge

$$\boxed{g_{\mu\nu} = a^2 \begin{pmatrix} 1+2\Psi & 0 \\ 0 & -(1-2\Phi)\delta_{ij} \end{pmatrix} \quad \text{NEWTONIAN GAUGE}}$$

- We consider only <u>scalar</u> metric perturbations:
  - Vector perturbations get suppresed during inflation.
  - Tensor perturbations <u>are</u> produced during inflation $\Rightarrow$ discussed in Section 4.7.

a) Conservation equations

$\nabla^\mu T_{\mu\nu} = 0$. (Also, $\nabla^\mu T^{(a)}_{\mu\nu} = 0$ if there is not energy-momentum transfer between different species)

- $\nu = 0 \Rightarrow$ <u>Continuity equation</u>:

$$\partial_\eta \delta\rho = \underbrace{-3\mathcal{H}(\delta\rho + \delta P)}_{\substack{\text{dilution effect} \\ \text{from background} \\ \text{expansion} \\ \text{"}\bar{\rho}' = -3\mathcal{H}(\bar{\rho}+\bar{P})\text{"}}} + \underbrace{3\Phi'(\bar{\rho}+\bar{P})}_{\substack{\text{effect of} \\ \text{local expansion} \\ \text{rate} \\ \text{"}(1-\Phi)\,a\text{"}}} - \partial_i \underset{\substack{\downarrow \\ \text{peculiar} \\ \text{velocity}}}{q^i}$$

- $\nu = i \Rightarrow$ Euler equation

$$\partial_\eta q^i = \underbrace{-4\mathcal{H} q^i}_{\substack{\text{dilution due} \\ \text{to background} \\ \text{expansion}}} \underbrace{- (\bar{\rho}+\bar{P})\partial^i\Psi - \partial^i\delta P - \partial_j \Pi^{ij}}_{\text{force terms}}$$

no force terms:

$$q \simeq \underbrace{(\bar{\rho}+\bar{P})}_{\propto \frac{1}{a^3}} \underbrace{v}_{\propto \frac{1}{a}}$$

(*) In terms of density contrast "δ_a" and velocity "v_i"

$$\boxed{\begin{aligned} \delta_a' &= -\left(1+\frac{\bar{P}_a}{\bar{\rho}_a}\right)\left(\partial_i v^i_a - 3\Phi'\right) - 3\mathcal{H}\left(\frac{\delta P_a}{\bar{\rho}_a} - \frac{\bar{P}_a}{\bar{\rho}_a}\delta_a\right) \qquad \text{Continuity eq.} \\ v^{i\,\prime}_a &= -\left(\mathcal{H}+\frac{\bar{P}_a'}{\bar{\rho}_a+\bar{P}_a}\right)v^i_a - \frac{1}{\bar{\rho}_a+\bar{P}_a}\left(\partial^i \delta P_a - \partial_j \Pi^{ij}_a\right) - \partial_i \Psi \qquad \text{Euler eq.} \end{aligned}}$$

Comment: Four scalar perturbations $(\delta_a, \delta P_a, v_a, \Pi_a)$,
but only two equations.

- If perfect fluid, strong interactions keep pressure isotropic: $\Pi_a = 0$. Also, $\delta P_a = \underset{\substack{\downarrow \\ \text{sound speed} \\ \text{of the fluid}}}{c_{s,a}^2}\, \delta\rho_a$

- Decoupled or weakly interacting species (e.g. neutrinos):
  $\Pi_a \neq 0$; $\delta P_a \neq c_{s,a}^2\, \delta\rho_a$ ⇒ Need to solve Boltzmann eqs. for the perturbed distribution function to close the system.

(*) Examples:

$\delta_m \sim$ const

$\delta_m \uparrow\uparrow$

- Clustering of dark matter:

  $P_m = 0$, $\Pi^{ij}_m = 0$

$$\Rightarrow \quad \delta_m'' + \underset{\substack{\uparrow \\ \text{friction}}}{\mathcal{H}\delta_m'} = \underset{\substack{\uparrow \\ \text{gravity}}}{\nabla^2\Psi} + 3\left(\Phi'' + \mathcal{H}\Phi'\right)$$

- Radiation fluctuations

  $P_r = \frac{1}{3}\rho_r$

  $\Pi^{ij}_r = 0$

$$\Rightarrow \quad \delta_r'' - \underset{\substack{\uparrow \\ \text{pressure}}}{\frac{1}{3}\nabla^2\delta_r} = \underset{\substack{\uparrow \\ \text{gravity}}}{\frac{4}{3}\nabla^2\Psi} + 4\Phi''$$

b) Einstein equations

$$\delta G^{\mu}{}_{\nu} = 8\pi G \delta T^{\mu}{}_{\nu}$$

- $\mu\nu = 00 \Rightarrow$ Relativistic Poisson equation:

$$\nabla^2 \Phi \underbrace{- 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)}_{\text{relativistic correction (relevant for } k \lesssim \mathcal{H})} = 4\pi G a^2 \delta\rho \quad \left[\delta\rho = \sum_a \delta\rho_a\right]$$

- $\mu\nu = jj$ (traceless part) $\Rightarrow$ $\Phi - \Psi = 8\pi G a^2 \Pi \quad \left[\Pi = \sum_a \Pi_a\right]$

  ↳ If perfect fluid $(\pi \simeq 0)$ $\Rightarrow \boxed{\Psi \simeq \Phi}$ $\begin{pmatrix} \text{- baryons} \\ \text{- DM} \\ \text{- photons} \end{pmatrix}$ ✓

- $\mu\nu = 0i \Rightarrow$ $\Phi' + \mathcal{H}\Phi = -4\pi G a^2 q$

- $\mu\nu = jj$ (trace part) $\Rightarrow$

$$\boxed{\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P \quad \text{(assuming } \Pi = 0)}$$

③.③ Solution for Φ:

$$\left\{\begin{array}{ll} \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P & (*) \\ \nabla^2 \Phi = -3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho & (**) \end{array}\right.$$

- **During MD era:**

$\delta P = 0$

$\mathcal{H} = \frac{2}{\eta} \rightarrow 2\mathcal{H}' + \mathcal{H}^2 = 0 \xrightarrow{(*)}$ $\boxed{\Phi'' + 3\mathcal{H}\Phi' = 0}$

$$\Phi(\eta) = C_1 + \overset{0}{\cancel{\frac{C_2}{\eta^5}}} \simeq \text{const}$$

- **During RD era:**

$\delta\rho = \frac{\delta P}{3}$ ⊙

$\mathcal{H} = \frac{1}{\eta} \rightarrow 2\mathcal{H}' + \mathcal{H}^2 = -\mathcal{H}^2 \xrightarrow{(*)}$ $\Phi'' + 3\mathcal{H}\Phi' - \mathcal{H}^2\Phi = 4\pi G a^2 \overset{⊙}{\delta P} = 4\pi G a^2 \frac{\delta\rho}{3} \overset{(**)}{=}$

$$= \frac{1}{3}\left(\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)\right)$$

$$\boxed{\Phi'' + 4\mathcal{H}\Phi' = \frac{1}{3}\nabla^2\Phi}$$

$$\Phi(\eta, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \Phi_k(\eta)\, e^{i\vec{k}\vec{x}} \Rightarrow \Phi_{\vec{k}}'' + \frac{4}{\eta}\Phi_{\vec{k}}' + \frac{1}{3}k^2 \Phi_{\vec{k}} = 0 \rightarrow$$

$$\rightarrow \boxed{\Phi_{\vec{k}} \simeq 3\underbrace{\Phi_{\vec{k}}(0)}_{\Phi_{\vec{k}}(0) = -\frac{2}{3}\mathcal{R}_k(0) \ \text{[see below]}}\left(\frac{\sin y - y\cos y}{y^3}\right) \quad ; \ y = \frac{1}{\sqrt{3}}k\eta}$$

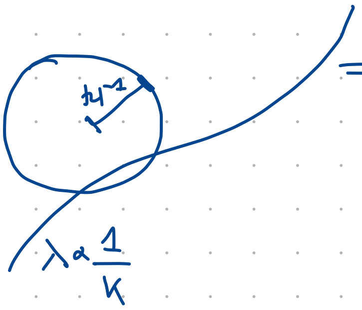
(RD)

$$\Phi_{\vec{k}}(\eta) = -2\mathcal{R}_k(0)\left(\frac{\sin y - y\cos y}{y^3}\right) \quad ; \quad y = \frac{1}{\sqrt{3}}k\eta$$

- During RD, two regimes:
  - Superhorizon perturbations:

$k \ll \mathcal{H}$

comoving Hubble radius



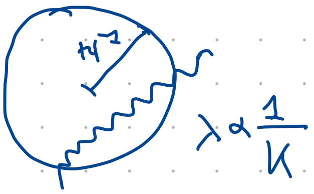
$$\Rightarrow \quad y \simeq \frac{1}{\sqrt{3}}\frac{k}{\mathcal{H}} \ll 1$$

$$\downarrow$$

$$\boxed{\Phi_{\vec{k}}(\eta) \simeq \text{const}}$$

- Subhorizon perturbations:

$k \gg \mathcal{H}$



$$\Rightarrow \quad y \simeq \frac{1}{\sqrt{3}}\frac{k}{\mathcal{H}} \gg 1$$

$$\downarrow$$

$$\boxed{\Phi_{\vec{k}}(\eta) \simeq -6\mathcal{R}_{\vec{k}}(0)\,\frac{\cos\left(\frac{1}{\sqrt{3}}k\eta\right)}{(k\eta)^2}}$$

Oscillations with frequency $\frac{1}{\sqrt{3}}k\eta$ and decaying amplitude $\eta^{-2} \propto a^{-2}$

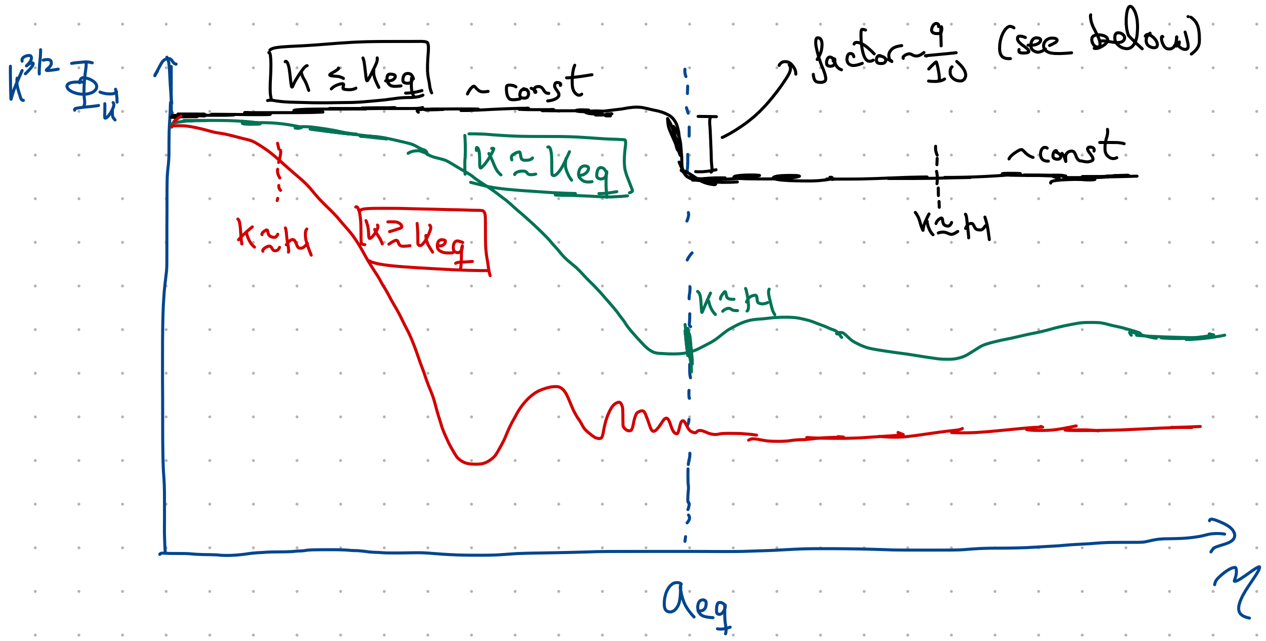
$$\boxed{\mathcal{H}\downarrow : \text{superhorizon} \rightarrow \text{subhorizon}}$$

$$\Phi_{\vec{k}}(\eta) = \begin{cases} -2\mathcal{R}_{\vec{k}}(0)\left(\frac{\sin y - y\cos y}{y^3}\right) & ;\ \text{during RD} \\ \text{const} & ;\ \text{during MD} \end{cases}$$

with $y = \frac{1}{\sqrt{3}} k\eta \simeq \frac{k}{\mathcal{H}}$

- Let us define $k_{eq} \equiv \mathcal{H}_{eq} = \mathcal{H}(\overset{\text{matter-radiation equality}}{\eta_{eq}})$ as the mode that enters the horizon at the matter-radiation equality.

  - $k \gtrsim k_{eq}$: mode enters horizon during RD stage.
  - $k \lesssim k_{eq}$: mode enters horizon during MD stage.



3.4. Initial conditions

- **Adiabatic fluctuations**: Fluctuations for which local state of matter (e.g. ρ, P) at some spacetime point $(\eta, \vec{x})$ of the perturbed universe, is the same as in the background universe at some slightly later time $\eta + \delta\eta(\vec{x})$:

$$\boxed{\bar{\rho}_a\left(\eta + \delta\eta(\vec{x})\right) = \bar{\rho}_a(\eta) + \delta\rho_a(\eta, \vec{x})} \quad \text{for all species "a"}$$

$$\downarrow$$

$$\delta\rho_a \simeq \bar{\rho}_a' \, \delta\underline{\eta(\vec{x})} \quad \text{→perturbations created by a local time shift}$$

$$\Big\downarrow \; \delta\eta \text{ the same for all species}$$

$$\text{for } \forall a,b \quad \frac{\delta\rho_a}{\rho_a'} = \frac{\delta\rho_b}{\rho_b'} \xrightarrow[\overset{\uparrow}{\bar{\rho}_a' = -3\mathcal{H}(1+w_a)\bar{\rho}_a}]{} \boxed{\frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b}}$$

$$\Big\downarrow \; w_m = 0 \; ; \; w_r = 1/3$$

$$\boxed{\delta_r = \frac{4}{3}\delta_m} \quad (*)$$

- **Initial conditions**

- All scales of interest are outside the Hubble radius $(k \ll \mathcal{H})$ at sufficient early times. We fix adiabatic initial conditions at this stage (given by $(*)$).

- Matter fluctuations hold $\delta_r = \frac{4}{3}\delta_m$ (for $a = \gamma, b, c$) for $k \ll \mathcal{H}$, but evolve differently after they enter the horizon $(k \gg \mathcal{H})$

- Conditions for Φ:

$$\underset{k \ll \mathcal{H}}{\cancel{\nabla^2}}\Phi \simeq 3\mathcal{H}(\underset{\Phi \sim \text{const}}{\cancel{\Phi'}} + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho \quad \Rightarrow \quad \Phi = -\frac{4\pi G a^2 \delta\rho}{3\mathcal{H}^2} \underset{\mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2}{=} \frac{1}{2}\frac{\delta\rho}{\rho} = \frac{1}{2}\delta \simeq \frac{1}{2}\delta_r$$

$$\Rightarrow \boxed{\delta \simeq \delta_r = -2\Phi = \text{const}}$$

3.5. Comoving curvature perturbation

In Newtonian gauge, it takes the form:

$$\boxed{\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\bar\rho + \bar p}\, q} \qquad \left(\delta T^0{}_j = q = \frac{\theta'}{\bar\rho + \bar p} \right)$$

- $\mathcal{R}$ is constant at superhubble scales, including when the equation of state changes:

$$\mathcal{R}' \xrightarrow{k \ll \mathcal{H}} 0 \quad \Rightarrow \quad \boxed{\mathcal{R} \overset{k \ll \mathcal{H}}{\simeq} \text{const}}$$

- $\mathcal{R} \overset{k \ll \mathcal{H}}{\simeq} -\dfrac{5+3w}{3+3w}\,\Phi \quad \Rightarrow \quad \Phi^{(MD)} \simeq \dfrac{9}{10}\, \Phi^{(RD)}$

→ exercise!

- $\mathcal{R}$ is gauge-invariant (although expression above is written in Newtonian gauge variables)

- $\mathcal{R}$ connects quantum fluctuations generated during inflation with later structure formation.

- Quantum mechanics only predicts statistics of initial conditions (correlations of $\mathcal{R}(\vec{x})$ in different directions).

  If initial conditions are Gaussian, they are completely specified by:

hom & iso

$$\boxed{\langle \mathcal{R}(\vec{x})\, \mathcal{R}(\vec{x}\,') \rangle \equiv \xi_R(\vec{x}, \vec{x}\,') \overset{!}{=} \xi_R(|\vec{x} - \vec{x}\,'|)}$$

$\Big\updownarrow$ Fourier transform

$$\boxed{\langle \mathcal{R}(\vec{k})\, \mathcal{R}^*(\vec{k}\,') \rangle \equiv \frac{2\pi^2}{k^3} \Delta_{\mathcal{R}}^2(k)\, \delta_D(\vec{k} - \vec{k}\,')}$$

Power spectrum

- Inflation predicts: $\Delta_{\mathcal{R}}^2(k) = A_s \left(\dfrac{k}{k_*} \right)^{n_s - 1}$

$A_s \simeq 2 \times 10^{-9}$

$n_s \simeq 0.96$

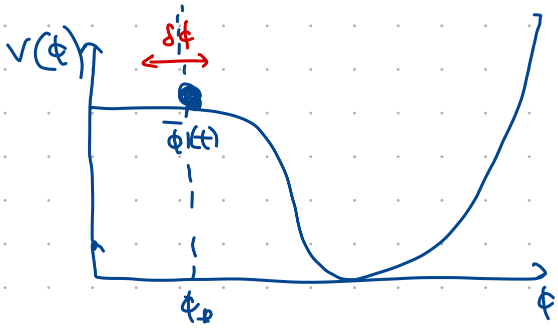
$k_* \simeq 0.05\,\text{Mpc}^{-1}$

4.5 Inflaton fluctuations: classical

- According to quantum fluctuations, the inflaton fluctuates:

$$\phi(\vec{x},t) = \bar{\phi}(t) + \delta\phi(t,\vec{x}) \qquad (\delta\phi << \bar{\phi})$$

- Therefore, inflation ends at slightly different times at different points:



$$\delta t(\vec{x}) \longleftrightarrow \delta\rho(t,\vec{x}) \longleftrightarrow \mathcal{R}(t,\vec{x})$$

- We study first the classical dynamics of these fluctuations:

$$S = \int d\eta \, d^3\vec{x} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu\phi \, \partial_\nu\phi - V(\phi) \right]$$

- We consider eq. of motion at linear order → we need to expand the action up to quadratic order, for both $\delta\phi$, $\delta g_{\mu\nu}$.

- It's complicated. Computation simplifies in spatially-flat gauge:

$$g_{ij} = -a^2 \delta_{ij} \quad \text{(unperturbed)}.$$

- Also, $\delta g_{0\mu} << \delta\phi$ for $\varepsilon \to 0$ ⇒ we only consider $\delta\phi$.

$$\boxed{\phi(\eta,\vec{x}) = \bar{\phi}(\eta) + \underbrace{\frac{f(\eta,\vec{x})}{a(\eta)}}_{\delta\phi(\eta,\vec{x})}}$$

⊕ Computation:

$$\text{FLRW} \overset{\sqrt{-g}=a^4}{\Longrightarrow} \quad S=\int d\eta\, d^3\vec{x}\left[\frac{1}{2}a^2\left(\phi'^2-(\nabla\phi)^2\right)-a^4V(\phi)\right]$$

$$S=S^{(0)}+\underset{\cancel{\int d^4\vec{x}\,\partial_\mu j^\mu}}{S^{(1)}}+S^{(2)}+\dots \qquad \phi(\eta,\vec{x})=\bar{\phi}(\eta)+\frac{\delta(\eta,\vec{x})}{a(\eta)}$$

$$\downarrow$$

$$S_{(2)}=\frac{1}{2}\int d\eta\, d^3\vec{x}\left[(\delta')^2-2\mathcal{H}\underbrace{\delta\delta'}+\mathcal{H}^2\delta^2-(\nabla\delta)^2-a^2V_{,\phi\phi}\delta^2\right]$$

$$"\mathcal{H}\partial_\eta(\delta^2)=\partial_\eta(\cancel{\mathcal{H}}\delta^2)-\mathcal{H}'\delta^2$$

$$\downarrow$$

$$S_{(2)}=\frac{1}{2}\int d\eta\, d^3\vec{x}\left[(\delta')^2-(\nabla\delta)^2+(\underbrace{\mathcal{H}'+\mathcal{H}^2}_{"a''/a}-a^2V_{,\phi\phi})\delta^2\right]$$

$$\downarrow$$

$$S_{(2)}=\frac{1}{2}\int d\eta\, d^3\vec{x}\left[(\delta')^2-(\nabla\delta)^2+\left(\frac{a''}{a}-a^2\cancel{V_{,\phi\phi}}\right)\delta^2\right]$$

$$\eta_V=m_p^2\frac{V''}{V}\simeq\frac{V''a}{\mathcal{H}}\simeq\frac{V''a}{a''/a}\ll 1$$

$$\downarrow$$

$$\boxed{S_{(2)}=\int d\eta\, d^3\vec{x}\left[\frac{1}{2}(\delta')^2-(\nabla\delta)^2+\frac{a''}{a}\delta^2\right]}$$

Minimizing from the action:

$$\boxed{\delta_k''+\left(k^2-\frac{a''}{a}\right)\delta_k=0} \qquad \delta_k(\eta)=\int\frac{d^3\vec{x}}{(2\pi)^{3/2}}\delta(\eta,\vec{x})e^{-i\vec{k}\vec{x}}$$

Mukhanov-Sasaki equation

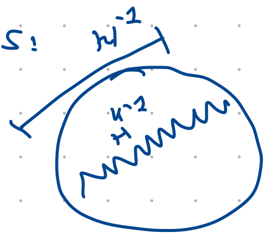
- Quasi-de Sitter: $\frac{a''}{a}\simeq 2\mathcal{H}^2=\frac{2}{\eta^2} \longrightarrow \boxed{\delta_k''+\left(k^2-\underset{\mathcal{H}^2}{\underbrace{\frac{2}{\eta^2}}}\right)\delta_k=0}$

- Also, comoving Hubble radius shrinks during inflation:

$$\mathcal{H}^{-1}=(aH)^{-1}\underset{a\simeq-\frac{1}{H\eta}\,;\ \eta\in[-\infty,0]}{=}-\eta$$

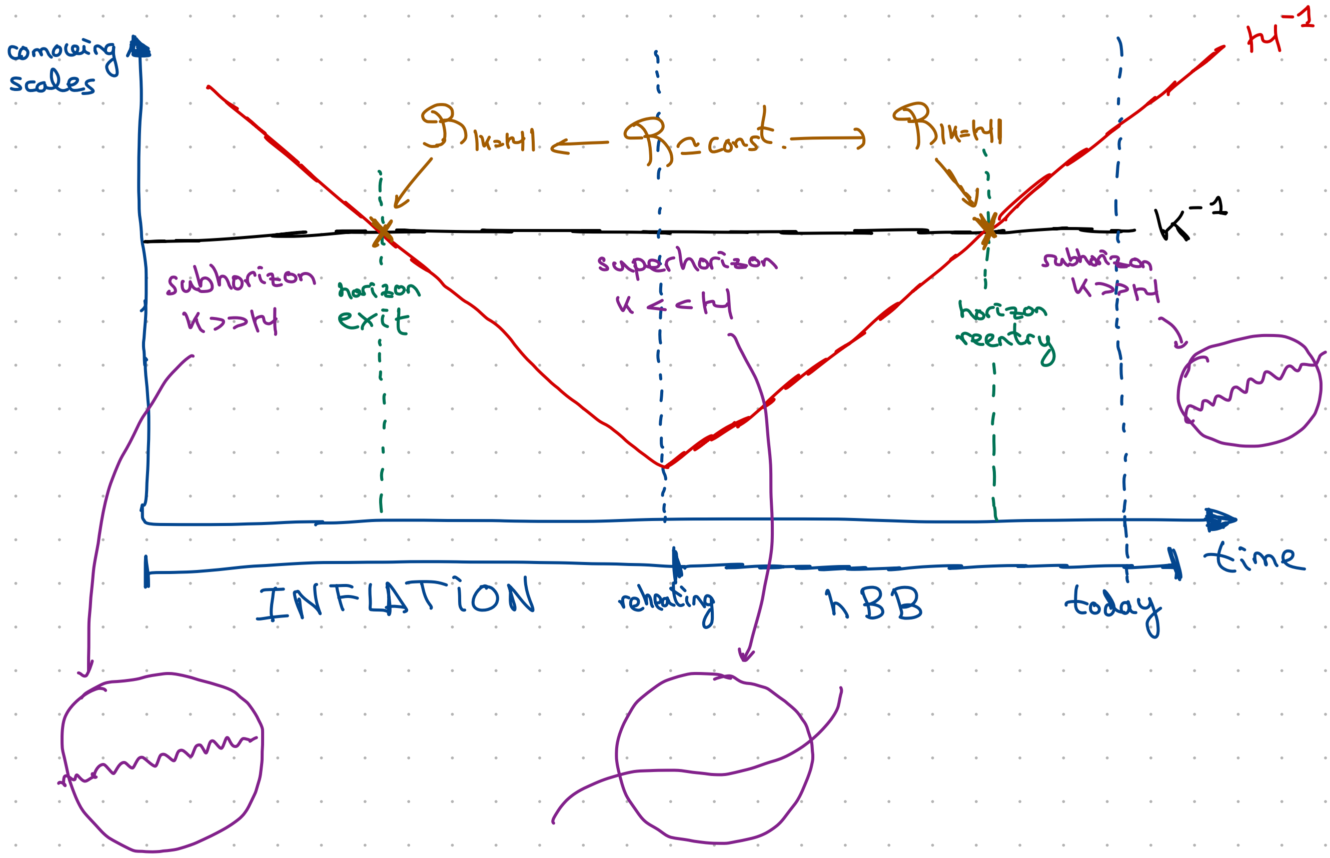
- Given a Fourier mode k, it is inside the Hubble radius at sufficient early times:

$\mathcal{H}^{-1}\sim|\eta|\gg k^{-1}$;



$$\Rightarrow \boxed{\begin{array}{c}\delta_k''+k^2\delta_k\simeq 0\\ [\text{for } |k\eta|\gg 1]\end{array}}$$

$(H^{-1} \sim -\eta)$ inflation



$\mathcal{R}$ is constant for superhorizon perturbations $\Rightarrow$
$\Rightarrow$ computing $\mathcal{R}_{|k=H|}$ for a given inflationary model provides the initial condition for the later structure formation.

We want to compute $\mathcal{R}_{|k=H|}$!

- → First we compute $\delta\phi_{|k=H|}$ (section 4.6); then $\mathcal{R}_{|k=H|}$ (section 4.7)
- → Similar for tensor perturbations / gravitational waves (section 4.7)

(4.6) Inflaton fluctuations: quantum

$$S_{(2)} = \int d\eta \, d^3\vec{x} \, \frac{1}{2} \left[(f')^2 - (\nabla f)^2 + \frac{a''}{a} f \right] \qquad f = a\delta\phi$$

- Momentum conjugate: $\pi = \frac{\partial \mathcal{L}}{\partial f'} = f'$

Quantization of the theory:

[Note: In Heisenberg picture: operators depend on time, states do not]

Step 1: We promote fields $\{f(\eta,\vec{x}), \pi(\eta,\vec{x})\}$, to quantum operators $\{\hat{f}(\eta,\vec{x}), \hat{\pi}(\eta,\vec{x})\}$, with commutation relation:

$$\boxed{[\hat{f}(\eta,\vec{x}), \hat{\pi}(\eta,\vec{x})] = i\,\delta_D(\vec{x}-\vec{x}')}$$

Dirac delta

$$[\hat{f}_{\vec{k}}(\eta), \hat{\pi}_{\vec{k}'}(\eta)] = \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} \int \frac{d^3\vec{x}'}{(2\pi)^{3/2}} \underbrace{[f(\eta,\vec{x}), \hat{\pi}(\eta,\vec{x})]}_{i\delta_D(\vec{x}-\vec{x}')} e^{-i\vec{k}\vec{x}} e^{-i\vec{k}'\vec{x}'} =$$

$$= i \int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i(\vec{k}+\vec{k}')\vec{x}} = \underset{(**)}{i\,\delta_D(\vec{k}+\vec{k}')} \Rightarrow$$ Modes with different wavelengths commute

Step 2: Mode expansion:

$$\underset{(*)}{\boxed{\hat{f}_{\vec{k}}(\eta) = f_k(\eta)\,\hat{a}_{\vec{k}} + f_k^*(\eta)\,a_{\vec{k}}^{+}}} \; ; \qquad \boxed{f_k'' + w_k^2(\eta) f_k = 0 \, ; \quad w_k^2(\eta) \equiv k^2 - \frac{a''}{a}}$$

creation and destruction operators

(MS eq.)

Step 3: Normalization

Substituting (*) into (**):

$$W[f_k] \times [a_{\vec{k}}, \hat{a}_{\vec{k}}^{+}] = \delta_D(\vec{k}+\vec{k}^{*}) \; ;$$ where $W[f_k] = -i(f_k, \dot{f}_k^* - \dot{f}_k f_k^*)$ is the Wronskian.

Choice: $W[f_k] = 1 \rightarrow \boxed{[\hat{a}_{\vec{k}}, a_{\vec{k}'}^{+}] = \delta_D(\vec{k}+\vec{k}')}$

Step 4: Vacuum state: $\boxed{\hat{a}_k |0\rangle = 0} \longrightarrow \hat{a}_k^{+} |0\rangle$ creates particle states.

Step 5: Choice of vacuum: The Mukhanov-Sasaki equation has two possible solutions, so we need to fix two constants.

- In Minkowski spacetime:

constants

$$f_k'' + k^2 f_k = 0 \longrightarrow f_k = \underline{\alpha_k} e^{-ik\eta} + \underline{\beta_k} e^{+ik\eta}$$

- Asking $\underset{\text{Hamiltonian}}{\hat{H}}|0\rangle = E|0\rangle$ with $E>0 \Rightarrow f_k \sim e^{-ik\eta}$

- From $W[f_k] = 1 \Longrightarrow f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta}$

- In generic time-dependent backgrounds, the procedure is ambiguous. However, for inflation there is a preferred choice: the Bunch-Davies vacuum:

  - At very early times, modes of cosmological interest were deep inside the horizon, $|k\eta| << 1$

$$\omega_k^2 = k^2 - \frac{a''}{a} \simeq k^2 - \frac{2}{\eta^2} \xrightarrow{\eta\to-\infty} k^2$$

$$f_k'' + k^2 f_k \simeq 0$$ (same as in Minkowski space) $\Longrightarrow f_k \propto e^{-ik\eta}$

  - In general:

$$f_k'' + \left(k^2 - \frac{2}{\eta^2}\right) f_k = 0$$

$W[f_k] = 1 \Rightarrow |\alpha|^2 - |\beta|^2 = 1$

$$\downarrow$$

$$f_k(\eta) = \alpha_k \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) + \beta_k \frac{e^{ik\eta}}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)$$

$$\Big\downarrow \; \alpha = 1 \,;\; \beta = 0$$

$$\boxed{f_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)}$$

Bunch-Davies vacuum

Zero-point fluctuations:

$$\hat{f}(\eta,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left(f_k(\eta)\,\hat{a}_{\vec{k}} + f_k^*(\eta)\, a^+_{\vec{k}} \right) e^{i\vec{k}\cdot\vec{x}}$$

$$\left[\hat{a}_k, \hat{a}^+_{k'}\right] = \delta_D(\vec{k}+\vec{k}') \quad ; \quad \hat{a}_k|0\rangle = 0 \quad ; \quad \langle 0|\hat{a}_k^+$$

- $\langle 0|\hat{f}|0\rangle = 0$

evaluated at $\vec{x}=0$

- $\langle 0|\hat{f}^+(\eta,\vec{0})\, \hat{f}(\eta,\vec{0})|0\rangle =$

$$= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}'}{(2\pi)^{3/2}} \langle 0|\left(f_k^*\hat{a}^+_{\vec{k}} + f_k\hat{a}_{\vec{k}}\right)\left(f_{k'}\hat{a}_{\vec{k}'} + f_{k'}^*\hat{a}^+_{\vec{k}'}\right)|0\rangle =$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}'}{(2\pi)^{3/2}} f_k(\eta)\, f_{k'}^*(\eta)\, \langle 0|\left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}\right]|0\rangle =$$

$$\overset{\shortparallel}{\delta_D(\vec{k}+\vec{k}')}\ \underbrace{\langle 0|0\rangle}_{1}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} |f_k(\eta)|^2 = \int d\ln k\, \frac{k^3}{2\pi^2} |f_k(\eta)|^2 \quad //$$

$$\boxed{\text{Variance of inflaton is non-zero due to vacuum fluctuations}}$$

$$\boxed{\Delta_f^2(k,\eta) \equiv \frac{k^3}{2\pi^2} |f_k(\eta)|^2}$$

(dimensionless)
power
spectrum

$$f_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right)$$

$\downarrow$

$a \simeq -\frac{1}{H\eta}$

$$|f_k(\eta)|^2 = \frac{1}{2k}\left(1 + \frac{1}{(k\eta)^2}\right) \simeq \frac{1}{2k}\left(1 + \frac{a^2H^2}{k^2}\right)$$

$$\hookrightarrow \boxed{\Delta_f^2(k,\eta) = \frac{k^2}{4\pi^2}\left(1 + \frac{a^2H^2}{k^2}\right)}$$

$$\Delta_f^2(k,\eta) \equiv \frac{k^3}{2\pi^2}|f_k(\eta)|^2 = \frac{k^2}{4\pi^2}\left(1+\frac{a^2H^2}{k^2}\right)$$

$$\delta\phi = \frac{f}{a}$$

$$\Delta_{\delta\phi}^2(k,\eta) \equiv \frac{k^3}{2\pi^2}|\delta\phi_k(\eta)|^2 \overset{\checkmark}{=} \frac{k^2}{4\pi^2 a^2}\left(1+\frac{a^2H^2}{k^2}\right)$$

$$\boxed{\Delta_{\delta\phi}^2(k,\eta) \equiv \left(\frac{H}{2\pi}\right)^2\left[1+\left(\frac{k}{aH}\right)^2\right]} \xrightarrow[\substack{\text{superhorizon} \\ k \ll H = (aH)}]{} \boxed{\Delta_{\delta\phi}^2 \overset{k\ll H}{\simeq} \left(\frac{H}{2\pi}\right)^2}$$

Approximation: We approximate the power spectrum at horizon crossing as the one at superhorizon scales (valid for a quasi-de Sitter space).

Therefore, we have our final result:

$$\boxed{\Delta_{\delta\phi}^2 \simeq \left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH}} \rightarrow \text{evaluated at different times for different modes}$$

- If inflation were perfect de Sitter, then $H \equiv$ const, and we would have a scale-invariant power spectrum (independent on k).
- In reality, inflation is "quasi-de Sitter", so $H = H(a)$ slowly varies with time during inflation. This generates a small deviation with respect scale-invariance.
- Different inflaton potentials → different $H = H(a)$ → → different power spectra.

4.7 Curvature perturbations and gravitational waves

a) Curvature perturbation

- At horizon crossing, we switch from $\delta\phi$ to $\mathcal{R}$:

$$\boxed{\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\bar{\rho}+\bar{P}}\,\delta q} \;; \quad \delta T^{0}{}_{j} = -\partial_j \delta q$$

- Spatially-flat gauge, $\delta g_{ij} = 0$ (unperturbed) $\rightarrow \Phi = 0$.

- $$\delta T^{0}{}_{j} = \bar{g}^{0\mu}\partial_\mu \phi\, \partial_j \delta\phi = \underset{1/a^2}{\bar{g}^{00}}\partial_0 \bar{\phi}\,\partial_j\delta\phi = \frac{\bar{\phi}'}{a^2}\partial_j\delta\phi$$

- $$\bar{\rho}+\bar{P} = \frac{1}{a^2}(\bar{\phi}')^2$$

conformal time
→ cosmic time

$$\mathcal{R} = \frac{-\mathcal{H}}{\bar{\phi}'}\delta\phi = -H\frac{\delta\phi}{\dot{\phi}}$$

$\downarrow$

slow-roll parameter

$$\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2}$$

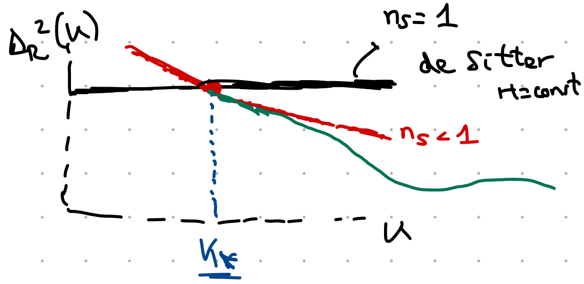
$$\Delta_{\mathcal{R}}^2(k) = \frac{H^2}{\dot{\phi}^2}\,\overset{= \left(\frac{H}{2\pi}\right)^2}{\Delta_{\delta\phi}^2(k)} = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\Bigg|_{k=aH} = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M_{pl}^2}\Bigg|_{k=aH}$$

$$\boxed{\Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M_{pl}^2}\Bigg|_{k=aH}}$$

(Note: H and ε depend on time)

$$\boxed{\Delta_R^2(k) = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{m_{pl}^2}\Bigg|_{k=aH}}$$

$\downarrow$



Linear approximation:

scalar tilt

$$\boxed{\Delta_R^2(k) \simeq A_s \left(\frac{k}{k_*}\right)^{n_s-1}}$$

scalar amplitude

pivot scale (to be fixed) → $k_* = 0.05\,\text{Mpc}^{-1}$

A_s and n_s are dimensionless

- $$A_s \simeq \Delta_R^2(k=k_*) = \frac{H^2}{8\pi^2\varepsilon m_{pl}^2}\Bigg|_{k=k_*} \overset{\uparrow}{=} \frac{1}{24\pi^2}\frac{1}{\epsilon_V}\frac{V}{m_{pl}^4}\Bigg|_{\phi=\phi_*}$$

$\parallel$ ← inflaton amplitude at which $k_* = aH_*$

$H^2 \simeq \frac{V(\phi)}{3m_{pl}^2}$; $\varepsilon \simeq \epsilon_V$

- $$n_s - 1 = \frac{d\ln\Delta_R^2}{d\ln k} = \frac{d\ln\Delta_R^2}{d\ln(aH)} \underset{\substack{\uparrow \\ H^2\simeq\text{const}}}{\simeq} \frac{d\ln\Delta_R^2}{d\ln a} = \frac{d\ln\Delta_R^2}{H\,dt} =$$

$$= \frac{1}{H}\frac{d\ln H^2}{dt} - \frac{1}{H}\frac{d\ln\varepsilon}{dt} = \frac{2\dot{H}}{H^2} - \frac{\dot{\varepsilon}}{\varepsilon H} \overset{\nearrow}{=} 2\varepsilon - \eta \overset{\uparrow}{=} -6\epsilon_V + 2\eta_V$$

$\varepsilon = \frac{\dot{H}}{H^2}$; $\eta = \frac{\dot{\varepsilon}}{H\varepsilon}$

$\varepsilon \simeq \epsilon_V$

$\eta \simeq 4\epsilon_V - 2\eta_V$

$$\boxed{\begin{aligned} A_s &= \frac{1}{24\pi^2}\frac{1}{\epsilon_V}\frac{V}{m_{pl}^4} \\ n_s - 1 &= -6\epsilon_V + 2\eta_V \end{aligned}}$$

- Constraints from Planck 2018:

$k_* = 0.05\,\text{Mpc}^{-1} \Rightarrow$ $A_s = (2.101^{+0.031}_{-0.034}) \times 10^{-9}$

$n_s = 0.965 \pm 0.004$ ↪ deviation from scale invariance!

b) Gravitational waves:

- Tensor perturbations are also produced during inflation:

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + 2\underline{h_{ij}})\,dx^i dx^j\right] \qquad h_{ij} \ll 1$$

- Compute the Einstein-Hilbert action up to second order:

$$S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\,R \Longrightarrow S_{(2)} = \frac{M_{pl}^2}{8}\int d\eta\, d^3\vec{x}\, a^2\left[(h_{ij}')^2 - (\nabla h_{ij})^2\right]$$

- "h_{ij}" has <u>two real degrees of freedom</u> → two polarizations:

$$h_{ij} = \frac{\sqrt{2}}{aM_{pl}}\begin{pmatrix} f_+ & f_\times & 0 \\ f_\times & -f_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \{f_+; f_-\}$$

$$S^{(2)} = \frac{1}{2}\sum_{\lambda=+,\times}\int d\eta\, d^3\vec{x}\left[(f_\lambda')^2 - (\nabla f_\lambda)^2 + \frac{a''}{a}f_\lambda^2\right]$$

two copies of the scalar action!

$$\Delta_h^2 = 2\times\left(\frac{2}{aM_{pl}}\right)^2\Delta_f^2 = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2}\Big|_{k=aH}$$

$$\Delta_f^2 = \left(\frac{H}{2\pi}\right)^2$$

$$\boxed{\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2}\Big|_{k=aH}}$$

→ only depends on H! Most model-independent and robust prediction of inflation.

↓

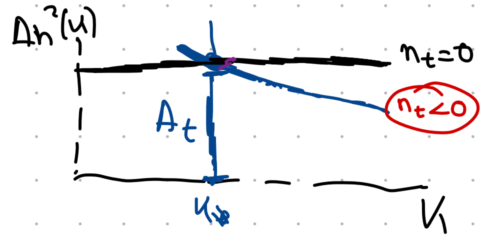
Linear approximation:

$$\boxed{\Delta_h^2(k) = A_t\left(\frac{k}{k_*}\right)^{n_t}}$$

tensor tilt

tensor amplitude

pivot scale ($k_* = 0.002\,Mpc^{-1}$)



Analogous computation to scalar perturbation shows:

$$\boxed{A_t = \frac{2V}{3\pi^2 M_{pl}^4} \quad ; \quad n_t = -2\epsilon_v}$$

- Results are quoted in terms of the tensor-to-scalar ratio:

$$\boxed{r = \frac{A_t}{A_s} = 16\,\epsilon_V} = -8n_t \qquad r = -8n_t$$

consistency check

$$A_s = \frac{V}{24\pi^2 \epsilon_V m_{pl}^4}$$

Upper bound from Planck 2018: $\boxed{r \lesssim 0.06}$

$(k_* = 0.002\,\mathrm{Mpc}^{-1})$

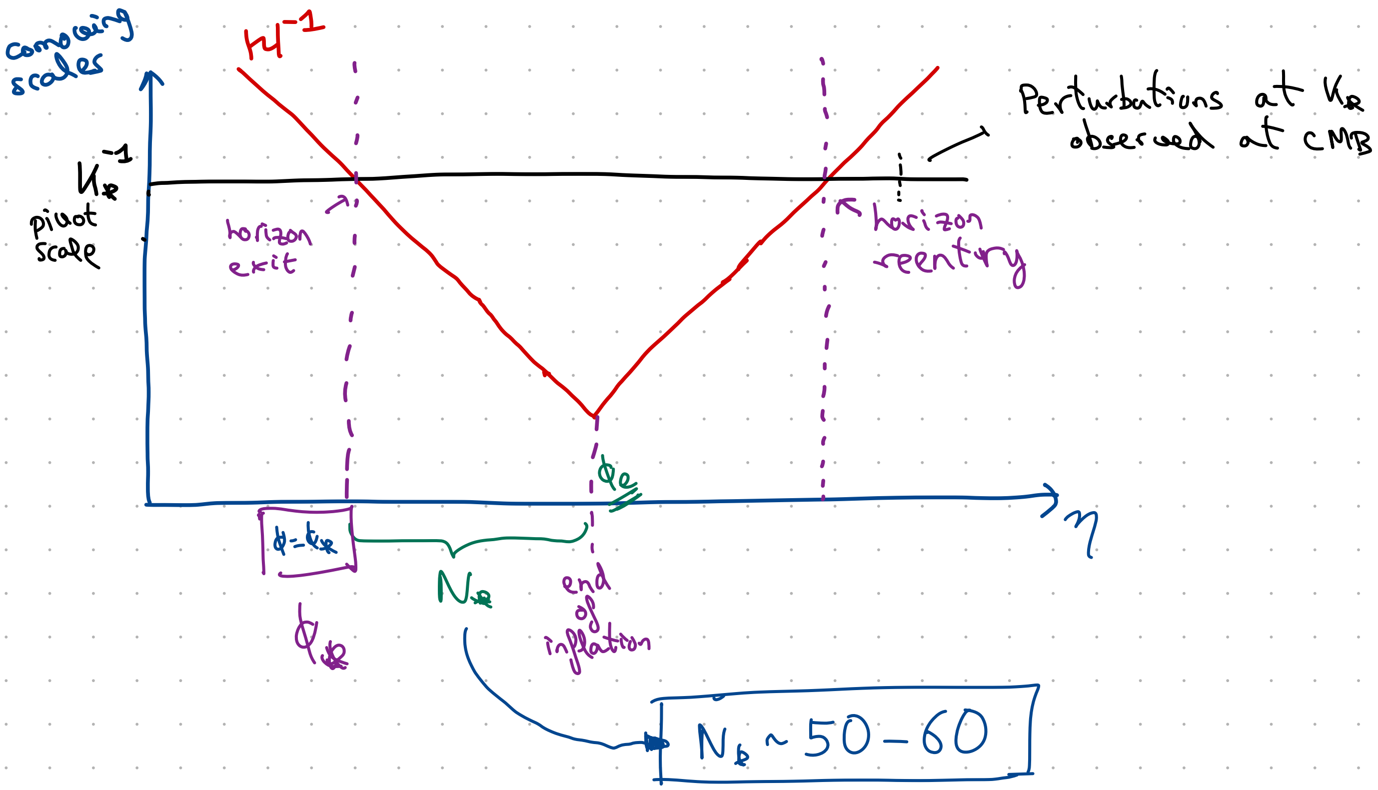
$$\Delta_R^2(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \longleftrightarrow A_s = \frac{V_*}{24\pi^2 \epsilon_{V*} m_{pl}^4}$$

$$n_s - 1 = -6\,\epsilon_{V*} + 2\,\eta_{V*}$$

$$\Delta_h^2(k) = A_t \left(\frac{k}{k_*}\right)^{n_t} \longleftrightarrow r = \frac{A_t}{A_s} = 16\,\epsilon_{V*}$$

$$n_t = -2\,\epsilon_{V*}$$

$\epsilon_{V*}, \eta_{V*}, V_*$: evaluated when k_* exits the horizon.



$$\boxed{N_* \sim 50 - 60}$$


Tensor-to-scalar ratio ($r_{0.002}$)
0.00
0.05
0.10
0.15
0.20
0.94
0.96
0.98
1.00
Primordial tilt (n_s)
Convex
Concave
TT,TE,EE+lowE+lensing
TT,TE,EE+lowE+lensing +BK14
TT,TE,EE+lowE+lensing +BK14+BAO
Natural inflation
Hilltop quartic model
α attractors
Power-law inflation
R^2 inflation
$V \propto \phi^2$
$V \propto \phi^{4/3}$
$V \propto \phi$
$V \propto \phi^{2/3}$
Low scale SB SUSY
N_*=50
N_*=60

Procedure to obtain prediction $\{n_s, r\}$ for a given inflationary model:

Given $V(\phi)$:

① Compute: $\epsilon_V(\phi) \equiv \frac{m_{pl}^2}{2}\left(\frac{V'}{V}\right)^2$, $\eta_V(\phi) \equiv m_{pl}^2 \frac{|V''|}{V}$

② End of inflation ϕ_e ; when $\epsilon_V(\phi_e) = 1$.

③ Obtain no. of e-folds before the end of inflation: $N(\phi) = \int_{\phi}^{\phi_e} \frac{|d\phi|}{\sqrt{2 m_{pl}^2 \epsilon_V(\phi)}}$ $\rightsquigarrow$ integral must always be positive!

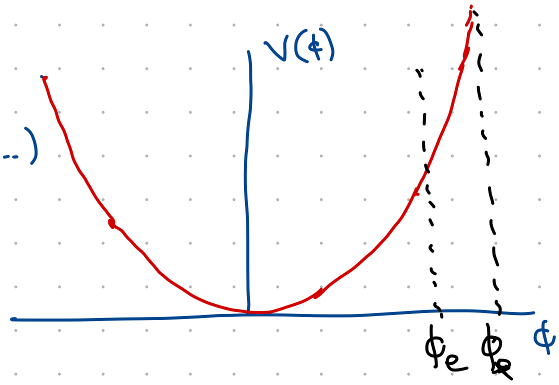
④ Obtain ϕ_* , such that $N(\phi_*) = N_*$ $(\simeq 50-60)$

↪ compute $\epsilon_{V*} \equiv \epsilon_V(\phi_*)$; $\eta_{V*} \equiv \eta_V(\phi_*)$;

[⑤ Use $A_s \simeq 2.3 \times 10^{-9}$ to fit potential parameters. This step is optional if ϵ_V, η_V do not depend on these.]

⑥ Compute n_s and r $\Rightarrow$ check compatibility with CMB bounds!

Example: $V(\phi) = \mu^{4-p}\phi^{p}$ $(p=2,4,6,\dots)$



① $V'(\phi) = p\,\mu^{4-p}\phi^{p-1}$

$V''(\phi) = p(p-1)\mu^{4-p}\phi^{p-2}$

$$\epsilon_V(\phi) = \frac{m_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{m_{pl}^2}{2}\left(\frac{p\cancel{\mu^{4-p}}\phi^{p-1}}{\cancel{\mu^{4-p}}\phi^{p}}\right)^2 = \frac{p\,m_{pl}^2}{2\phi^2}$$

$$\eta_V(\phi) = m_{pl}^2\frac{|V''|}{V} = \frac{p(p-1)m_{pl}^2}{2\phi^2}$$

② End of inflation: $\epsilon_V(\phi_e) = 1 \rightarrow \phi_e = \frac{p}{\sqrt{2}}m_{pl}$

③ $$N(\phi) = \int_{\phi_e}^{\phi}\frac{d\phi'}{m_{pl}}\frac{1}{\sqrt{2\epsilon_V(\phi')}} = \frac{1}{2m_p^2 p}\int_{\phi_e}^{\phi}\phi'\,d\phi' = \frac{1}{2pm_p^2}\left(\phi^2-\phi_e^2\right) =$$

$$= \frac{\phi^2}{2pm_{pl}^2} - \frac{p}{4}$$

④ $N(\phi_*) \simeq N_*\ (\simeq 50-60) \longrightarrow \phi_* = \sqrt{2p}\,m_{pl}\sqrt{N_*+\cancel{\frac{p}{4}}} \simeq \sqrt{2pN_*}\,m_{pl}$

$\epsilon_{V*} = \frac{p}{4N_*}$; $\eta_{V*} = \frac{p-1}{2}\frac{1}{N_*}$;

⑤ $V_* = \mu^{4-p}\phi_*^{p} = (2pN_*)^{p/2}\mu^{4-p}m_{pl}^{p}$

$$A_s = \frac{V_*}{24\pi^2\epsilon_{V*}m_{pl}^4} = \frac{2^{p/2}p^{\frac{p}{2}-1}}{6\pi^2}\mu^{4-p}m_p^{p-4}N_*^{\frac{p}{2}+1} \simeq 2.3\times10^{-9}$$

$$\boxed{\mu^{4-p} = 6\pi^2 2^{-p/2}p^{1-\frac{p}{2}}m_p^{4-p}N_*^{-1-\frac{p}{2}}A_s}$$

⑥ $$\boxed{\begin{aligned} n_s - 1 &= -6\epsilon_{V*} + 2\eta_{V*} = \frac{-2p}{2N_*} \\ r &= 16\epsilon_{V*} = \frac{4p}{N_*} \end{aligned}}$$

p=2: $(\mu/m_{pl})^2 = (1.9-2.7)\times10^{-11}$

$n_s = 0.960 - 0.967$

$r = 0.16 - 0.13$

p=4: $(\mu/m_{pl})^0 = (3.8-6.9)\times10^{-14}$

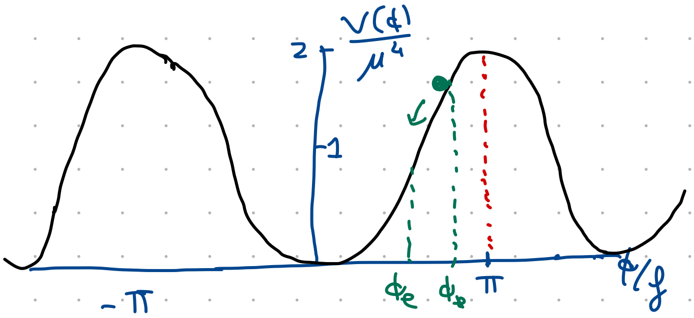
$n_s = 0.92 - 0.93$

$r = 0.32 - 0.27$

(for $N_* \simeq 50-60$)

Example: AXION POTENTIAL

$$V(\phi) = \mu^4 \left[1 - \cos\left(\frac{\phi}{f}\right) \right]$$



In Baumann lectures

$$\alpha \equiv \frac{M_{pl}^2}{f^2}$$

$$\boxed{n_s - 1 = - \frac{\alpha e^{N_* \alpha} + 1}{e^{N_* \alpha} - 1}} \; ; \; \boxed{r = 16 \epsilon_{V*} = 8\alpha \cdot \frac{1}{e^{N_* \alpha} - 1}}$$