Exercises for Theoretical Cosmology

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Example exercises

Example 4.1: Scalar field theory in the FLRW universe

The Lagrangian for a scalar field in curved spacetime is given by:

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \tag{1}$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor.

(a) From the Euler-Lagrange equation determine the equation of motion for the homogeneous scalar field $\phi(t)$ in a flat FLRW background.

(b) Now, consider a potential of the form $V(\phi) = m^2\phi^2/2$. Doing a transformation $\phi(t) = a^{-3/2}(t)\chi(t)$ show that the equation of motion can be written as:

$$\ddot{\chi} + \left(m^2 - \frac{3}{2}\dot{H} - \frac{9}{4}H^2\right)\chi = 0\,. \tag{2}$$

Find the solution of ϕ for the case $m^2 \gg H^2 \sim \dot{H}$ and determine the evolution of the energy density ρ_ϕ during the oscillatory phase.

Example 4.2: Reheating after Chaotic Inflation

After inflation the universe has cooled down and everything but the inflaton field ϕ has diluted. On the other hand, we have learned that during the hot big bang - described by the standard model of cosmology - the universe has been in a very hot and dense state where many particle species were present. So if this stage was preceded by inflation there must have been a transition phase in between, where the universe heated up. This phase is usually referred to as reheating.

Consider the following potential with an additional interaction term apart from the inflaton potential:

$$V(\phi,\chi) = V_{\text{inf}}(\phi) + V_{\text{reh}}(\phi,\chi) = \frac{1}{2}m_\phi^2\phi^2 + \lambda\phi\chi^2. \tag{3}$$

We will learn in the lecture that $m_\phi \simeq 6\times10^{-6}m_{\text{pl}}$. The equation of motion during the time of reheating is given by:

$$\ddot{\phi} + (3H + \Gamma_\phi)\dot{\phi} + V'(\phi) = 0\,, \tag{4}$$

where an additional decay term is present now. The inflaton decays into pairs of massless χ-particles and the corresponding decay rate Γ_ϕ is given by:

$$\Gamma_\phi = \frac{\lambda^2}{8\pi m_\phi}. \tag{5}$$

(a) Determine the field value of the inflaton at the end of inflation.

(b) Solve Eq. (8) numerically for the first few oscillations. How does the equation of state w evolve and what is its average value $\langle w \rangle$?
Hint: It is more convenient to solve the system of equations in terms of a new time variable $z = m_\phi t$. In the early phase the decay term and $V_{\rm reh}$ can be neglected.

(c) Show that the equation of motion can be approximated by the following Boltzmann equation:

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi. \tag{6}$$

How does the Boltzmann equation for the χ-field look like? Solve these two equations together with the first Friedmann equation numerically and describe how the energy densities evolve.
Hint: Note that the total energy density of the system needs to obey the continuity equation. Choose your favourite value of λ and initiate the system of equations shortly before the decay of ϕ at $H \sim 10 \times \Gamma_\phi$.

(d) Estimate the reheating temperature $T_{\rm RH}$ as a function of the coupling constant λ and compare it to the temperature at the electroweak phase transition $T_{\rm EW} \sim 100\text{GeV}$.

Hint: You may need the Virial theorem, which states that for a harmonically oscillating field: $\rho_\phi/2 = \langle \dot{\phi}^2/2 \rangle = \langle V(\phi) \rangle$.

Homework exercises

<u>Exercise 4.1:</u> Perturbed Einstein equations

In the first exercise sheet we have derived the Friedmann equations. Return to the *Mathematica* notebook we have started in exercise 1.2 and derive the Einstein equations for scalar perturbations in an FLRW universe filled with a perfect fluid.
Hint: Choose your favourite gauge, work in conformal time and recall that the energy momentum tensor for a perfect fluid is $T_{\mu\nu} = (\rho + P)U_\mu U_\nu - Pg_{\mu\nu}$, where the four-velocity in the Newtonian gauge is for example given by $U^\mu = ((1-\Phi), v^i)/a(\eta)$.

<u>Exercise 4.2:</u> Gravitational waves

The FLRW metric with tensor perturbations in conformal time is given by:

$$ds^2 = a^2(\eta)\Big(d\eta^2 - (\delta_{ij} + 2h_{ij})dx^i dx^j\Big)\,, \tag{7}$$

where h_{ij} is symmetric, traceless and transverse. The perturbation of the stress energy tensor are $\delta T_{ij} = 2a^2\bar{P}h_{ij} - a^2\Pi_{ij}$.

(a) Derive the perturbed Einstein equation with the *Mathematica* notebook from exercise 4.1. With $2\mathcal{H}' + \mathcal{H}^2 = -8\pi G a^2 \bar{P}$ you should find that:

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -8\pi G a^2 \Pi_{ij}\,. \tag{8}$$

(b) For a matter dominated universe and the assumption that anisotropic stress can be ignored ($\Pi_{ij} = 0$) the solution of the equation of motion above in Fourier space is:

$$h_{ij} \propto \frac{k\eta\cos(k\eta) - \sin(k\eta)}{(k\eta)^3}\,. \tag{9}$$

Show that on super-Hubble scale the solution tends to a constant value, whereas inside the Hubble radius it oscillates. Furthermore discuss how the amplitude evolves with the scale factor a.