Exercises for Theoretical Cosmology

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Example exercises

Example 3.1: Boltzmann equation

For the proper treatment of thermodynamical processes beyond equilibrium − such as the decoupling of species − one has to study the evolution of the phase space distribution function $f(x^\mu, p^\mu)$, which is governed by the so-called *Boltzmann equation*:

$$\hat{L}[f(x^\mu, p^\mu)] = C[f(x^\mu, p^\mu)]\,, \tag{1}$$

where $\hat{L}$ represents the *Liouville operator* and C the *collision term.* First we discuss the Liouville operator:

(a) Starting from $df/d\lambda = C$, where λ denotes the considered spacetime trajectory, show that the relativistic form of the Liouville operator is given by:

$$\hat{L} = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu}\,. \tag{2}$$

In general this can be a very elaborate and complicated expression. In the FLRW-model we are assuming homogeneity and isotropy, which leads to a substantial simplification of $\hat{L}$. Not only $\Gamma^\alpha_{\beta\gamma}$ obtains a fairly simple form, but also the dependency of the distribution function is reduced to the magnitude of the momentum $p = |\vec{p}|$ and time, i.e. $f(p,t)$ (or equivalently $f(E,t)$). Furthermore, from now on we assume the curvature parameter to be $k = 0$. The resulting Liouville operator is then:

$$\hat{L}[f(p,t)] = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a} pE \frac{\partial f}{\partial p}\,. \tag{3}$$

Small exercise for you: Use the *Mathematica* notebook from sheet 1 to compute the Liouville operator for a FLRW universe.

(b) Recall the definition of the number density n from the lecture. Using the $\hat{L}[f(p,t)]$ from above, show that the Boltzmann equation can be written as:

$$\frac{d}{dt} n + 3Hn = \frac{g}{(2\pi)^3} \int d^3p \frac{C[f]}{E}\,. \tag{4}$$

(c) Explain why the left-hand side of the Boltzmann equations for energy densities looks like:

$$\frac{d}{dt}\rho + 3H(\rho + P) = \frac{g}{(2\pi)^3} \int d^3p\, C[f]\,. \tag{5}$$

The collision term on the right-hand side of the Boltzmann equation is a little more involving (Note that if the collision term in Eq. (5) is zero, we just obtain the continuity equation). In principle an arbitrary number of particles may appear in the collision, but usually we only encounter processes involving three or four particles. In the following, we consider as a specific example the following process $\psi\bar{\psi} \to X\bar{X}$. For such a process the collision term has the following form:

$$\frac{g}{(2\pi)^3}\int d^3p_\psi \frac{C[f]}{E_\psi} = -\int d\Pi_\psi d\Pi_{\bar{\psi}} d\Pi_X d\Pi_{\bar{X}} (2\pi)^4 \delta^4(p_\psi + p_{\bar{\psi}} - p_X - p_{\bar{X}}) \times ...$$
$$... \times (|\mathcal{M}|^2_{\psi\bar{\psi}\to X\bar{X}} f_\psi f_{\bar{\psi}} (1 \pm f_X)(1 \pm f_{\bar{X}}) - |\mathcal{M}|^2_{X\bar{X}\to\psi\bar{\psi}} f_X f_{\bar{X}} (1 \pm f_\psi)(1 \pm f_{\bar{\psi}}))\,, \tag{6}$$

where

$$d\Pi_i = \frac{g}{(2\pi)^3}\frac{d^3p_i}{2E_i}\,. \tag{7}$$

This expression can be simplified by several assumptions. First, T (or CP) invariance implies that

$$|\mathcal{M}|^2_{\psi\bar{\psi}\to X\bar{X}} = |\mathcal{M}|^2_{X\bar{X}\to\psi\bar{\psi}} = |\mathcal{M}|^2\,. \tag{8}$$

Second, we use Maxwell-Boltzmann instead of Fermi-Dirac or Bose-Einstein distributions (which becomes exact in the limit $T/m \to 0$) and that the blocking or stimulated emission factors can be approximated by $(1 \pm f_i) \approx 1$.

(d) Assume further that the decay products are in thermal equilibrium and that the chemical potentials can be neglected, show that the collision term can be written as:

$$\int d\Pi_\psi d\Pi_{\bar{\psi}} d\Pi_X d\Pi_{\bar{X}} (2\pi)^4 \delta^4(p_\psi + p_{\bar{\psi}} - p_X - p_{\bar{X}}) |\mathcal{M}|^2 (f_\psi f_{\bar{\psi}} - f_\psi^{\text{eq}} f_{\bar{\psi}}^{\text{eq}})\,. \tag{9}$$

(e) By introducing the thermally averaged annihilation cross section times velocity and assuming that $n_\psi = n_{\bar{\psi}}$, show that the Boltzmann equation for ψ can be written as:

$$\frac{d}{dt} n_\psi + 3H\, n_\psi = -\langle\, \sigma |v|\, \rangle (n_\psi^2 - (n_\psi^{\text{eq}})^2)\,. \tag{10}$$

Note: If the particle ψ is underlying more than one process, the right hand-side of the Boltzmann equation will be a sum of several terms.

Remark: A more detailed derivation of the Boltzmann equation, in particular the collision term, can be found for example in Kolb/Turners *The Early Universe* p. 115 – 120. Also: note that we have only discussed the homogeneous and isotropic case. The equation becomes more complicated if the distribution functions also depend on $\vec{x}$ and $\vec{p}$. See for example *Cosmology* from Steven Weinberg.

Homework exercises

Exercise 3.1: Electron freeze-out

(a) Derive the Boltzmann equation for the free electron fraction $X_e \equiv n_e/n_b$:

$$\frac{d}{dx}X_e = -\frac{\lambda}{x^2}\left(X_e^2 - (X_e^{\rm eq})^2\right), \tag{11}$$

where $x \equiv B_{\rm H}/T$ contains the binding energy $B_{\rm H} = m_p + m_e - m_{\rm H} = 13.6\text{eV}$ and the temperature T. For the thermally averaged recombination cross section $\langle\sigma v\rangle$ the following approximation can be used:

$$\langle\sigma v\rangle = \sigma_T\left(\frac{B_{\rm H}}{T}\right)^{1/2}, \tag{12}$$

where $\sigma_T \simeq 2\times 10^{-3}\,\text{MeV}^{-2}$ is the Thomson cross section. The parameter λ should have the form:

$$\lambda \simeq 7\times 10^4\left(\frac{\Omega_b h}{0.035}\right). \tag{13}$$

(b) Solve the Boltzmann equation in *Mathematica* numerically and compare it to the *Saha equation* given in the lecture.

(c) Show that the freeze-out abundance is roughly given by $X_e^\infty \simeq x_f/\lambda$, where x_f denotes the freeze out time.

Remark: The baryon asymmetry of the universe is given by $\eta = n_B/n_\gamma \simeq 6\times 10^{-10}$.

Exercise 3.2: Transparency of the universe

The universe becomes transparent when the mean free path λ_γ becomes larger than the Hubble length $l_H = 1/H$. The mean free path of photons is mainly determined by scattering with electrons, thus given by $\lambda_\gamma \simeq 1/(\sigma_T n_e)$, where σ_T is the Thomson cross section and $n_e = X_e\, n_e^{tot}$ the number of free electrons. We have seen in the last exercise that X_e falls during recombination (which takes place between $z = 1300$ and 1000) from $X_e = 1$ to roughly 10^{-4}.

(a) Show that the universe is transparent after recombination, while it was opaque before.

(b) Determine the decoupling temperature.

Hint: Assume that the universe is solely filled with matter, i.e. $\Omega_m^0 = 1$, and go back to sheet 1 for inspiration, if needed. Also, remember charge neutrality of the universe and we can assume that roughly $n_B \simeq n_e^{tot}$. Use the Saha equation to estimate n_e in the second exercise.

Exercise 3.3: Black Body Forever!

The number density of photons per frequency interval during the time when they are in equilibrium with matter is given by the black-body spectrum:

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}. \tag{14}$$

After recombination photons have decoupled and move freely through space. Show that the photons maintain their black-body spectrum until today. Assume for simplification that the photon decoupling happened instantaneously.