Theoretical Cosmology Exam

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The exam is divided in two parts: **theory questions** (60 points) and **problems** (50 points). You need at least 50 points to pass the exam. The exam can be answered in English or German.

1 Theory questions

The following questions must be answered **shortly**, using few sentences.
Each question is worth *5 points*.

(1) State the Cosmological Principle. Is it valid at all length scales? Explain your answer.

(2) Write the different elements of the *cosmic inventory* in the ΛCDM model, together with their equation of state and their approximate relative contribution to the current total energy density (in percentage).

(3) Sketch a diagram of the luminosity distance $d_l(z)$, metric distance $d_m(z)$, and angular distance $d_a(z)$ as a function of redshift, for our own universe.

(4) Our Universe is approximately $14 \cdot 10^9$ years old. However, the most distant galaxy ever observed (GN-z11) is approximately $32 \cdot 10^9$ light-years from us. Explain why these two facts are not inconsistent.

(5) Compute the number of relativistic degrees of freedom g_* at $T = 1\,\mathrm{MeV}$.

Hint: The only SM particles with mass $m < 1\,\mathrm{MeV}$ are: electrons and positrons (e^+, e^-), neutrinos (ν_e, ν_μ, ν_τ), and photons (γ). All of them are in thermal equilibrium. Note that each of these particles has two possible spin states.

(6) Explain what is the *dark matter freeze-out* and how it relates with the so-called 'WIMP miracle'.

(7) Explain why the observed isotropy of the Cosmic Microwave Background constitutes an example of the *horizon problem* of classical cosmology.

(8) Write three different definitions of inflation.

(9) The equations of motion for a homogeneous scalar field ϕ with potential energy $V(\phi)$ evolving in a flat FLRW spacetime are

$$H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \ , \quad \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \ ,$$

where $H \equiv H(t)$ is the Hubble parameter. Write these equations in the slow-roll approximation by neglecting the appropriate terms.

(10) Write the perturbed FLRW metric in the Newtonian gauge in terms of the two Baumann potentials (consider only scalar perturbations). What condition do these functions obey if the expansion is sourced by a perfect fluid?

(11) Consider a perturbation with wave mode k. Write the condition that it must obey for it to be i) subhorizon, and ii) superhorizon.

(12) Consider a perturbation that is initially subhorizon, and consider its evolution since the beginning of inflation until today. With the help of a diagram, explain at which stages of cosmic history is the perturbation subhorizon/superhorizon, as well as indicate the times of *horizon exit* and *horizon reentry*.

2 Problems

Problem 2.1: Friedmann equations for a universe with matter and radiation *(25 points)*. Consider a spatially-flat universe composed of matter and radiation only. Denote their energy densities today as $\rho_r^{(0)}$ and $\rho_m^{(0)}$ respectively, and set the scale factor today to $a_0 = 1$.

(a) Show that the scale factor at the time of the *matter-radiation equality* (when the energy densities of both components are the same) is

$$a_{\rm eq} = \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \ .$$

(b) Consider the Friedmann equations for a spatially-flat FLRW Universe in cosmic time,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \ , \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ .$$

Show that these can be written in conformal time as

$$a'^2 = \frac{8\pi G}{3}\rho a^4 \ , \tag{1}$$

$$a'' = \frac{4\pi G}{3}(\rho - 3p)a^3 \ . \tag{2}$$

(c) Solve Eq. (2) for the universe with matter and radiation, and show that its solution can be written as

$$a(\eta) = \frac{2\pi G}{3}\rho_m^{(0)}\eta^2 + A\eta \ , \tag{3}$$

where A is an integration constant, and the initial condition $a(\eta = 0) = 0$ has been imposed.

(d) Fix the integration constant A as a function of $\rho_m^{(0)}$ and $a_{\rm eq}$ using Eq. (1).

(e) Show how the solution behaves in the cases of a universe with i) only radiation, and ii) only matter.

Problem 2.2: Hilltop-like inflation *(25 points).*

Consider the following *hilltop*-like inflationary potential,

$$V(\phi) = V_0 - \frac{\lambda\phi^4}{4} + \frac{\phi^6}{6v^2} \,,$$

where λ is dimensionless and v has dimensions of mass. We will assume that the inflaton amplitude is positive ($\phi > 0$). The potential shows a *plateau* $V(\phi) \sim V_0$ at small field amplitudes, a minimum at a certain amplitude $\phi = \phi_{\rm min}$, and a power-law behaviour $V(\phi) \sim \phi^6$ at large amplitudes.

(a) In order to avoid a cosmological constant, the potential must obey $V(\phi_{\rm min}) = 0$ at the minimum. Show that this condition implies the following relation between the parameters of the potential,

$$v = \left(\frac{12V_0}{\lambda^3}\right)^{1/4} \,.$$

(b) Show that in the limit $\phi \to 0$, the slow-roll parameters are given by

$$\epsilon_{\rm V}(\phi) \simeq \frac{m_p^2\lambda^2}{2V_0^2}\phi^6 \,, \quad \eta_{\rm V}(\phi) \simeq -\frac{3\lambda m_p^2}{V_0}\phi^2 \,, \tag{4}$$

and that inflation can then happen when the inflaton is over the plateau. Compute also the inflaton amplitude at the end of inflation, defined as $|\eta_{\rm V}(\phi_e)| = 1$ [with $\eta_{\rm V}$ given by Eq. (4)].

(c) Compute the slow-roll predictions for the spectral index n_s and the tensor-to-scalar ratio r. Write the results in terms of V_0, λ and N_*.

Hint: In order to simplify the computation, use the approximated expressions for the slow-roll parameters given in Eq. (4).

(d) Find the constraint that V_0 and λ must obey in order to explain the observed amplitude of the scalar perturbations, $A_s = 2.3 \times 10^{-9}$.

Hint: Recall that

$$\epsilon_{\rm V} = \frac{m_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta_{\rm V} = m_{\rm Pl}^2\frac{V''}{V}, \quad N(\phi) = \int \frac{|d\phi|}{\sqrt{2m_{\rm Pl}^2\epsilon_{\rm V}(\phi)}}$$

$$r = 16\epsilon_{{\rm V}*}, \quad n_s = 1 - 6\epsilon_{{\rm V}*} + 2\eta_{{\rm V}*}, \quad A_s = \frac{V_*}{24\pi^2 m_{\rm Pl}^4\epsilon_{{\rm V}*}}$$